S-72.245 Transmission Methods in Telecommunication Systems (4 cr)

Exponential Carrier Wave Modulation

Exponential modulation: Frequency (FM) and phase (PM) modulation

- Waveforms
  - Instantaneous frequency and phase
- Spectral properties
  - narrow band with arbitrary modulating waveform shape
  - wideband tone modulation
  - transmission BW
- Modulation of VCO, narrow band mixer
- Demodulation - FM/AM conversion, slope detector, balanced discriminator
  - Effect of additive interference in demodulation
- Preemphases and deemphases filters

Linear and exponential modulation

$$x_c(t) = A(t)\,\mathrm{Re}[\exp(\omega_c t + \phi(t))]$$

- In **linear** CW (carrier wave) modulation:
  - transmitted spectra resembles modulating spectra
  - spectral width does not exceed twice the modulating spectral width
  - destination SNR can not be better than the baseband transmission SNR (lecture: Noise in CW systems)
- In **exponential** CW modulation (FM/AM):
  - usually transmission BW>>baseband BW
  - bandwidth-power trade-off (channel adaptation): destination SNR can be much better than transmission SNR when transmission BW increased
  - baseband and transmitted spectra does not carry a simple relationship

Phase modulation (PM)

- Carrier Wave (CW) signal: $x_C(t) = A_C \cos(\underbrace{\omega_C t + \phi(t)}_{\theta_C(t)})$

- In exponential modulation the modulation is "in the exponent" or "in the angle"

$$x_c(t) = A_c \cos(\theta_c(t)) = A_c \,\mathrm{Re}[\exp(j\theta_c(t))]$$

- Note that in exponential modulation superposition does not apply:

$$x_c(t) = A\cos\{\omega_c t + k_f[a_1(t) + a_2(t)]\}$$

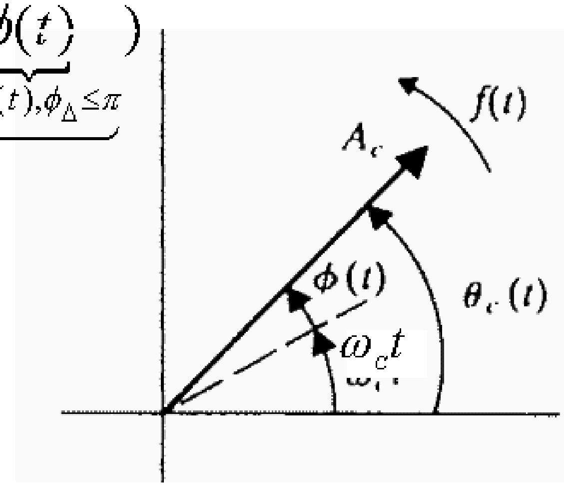
$$\neq A\cos\omega_c t + A\cos k_f[a_1(t) + a_2(t)]$$

- In **phase modulation** (PM) carrier phase is linearly proportional to the modulation amplitude:

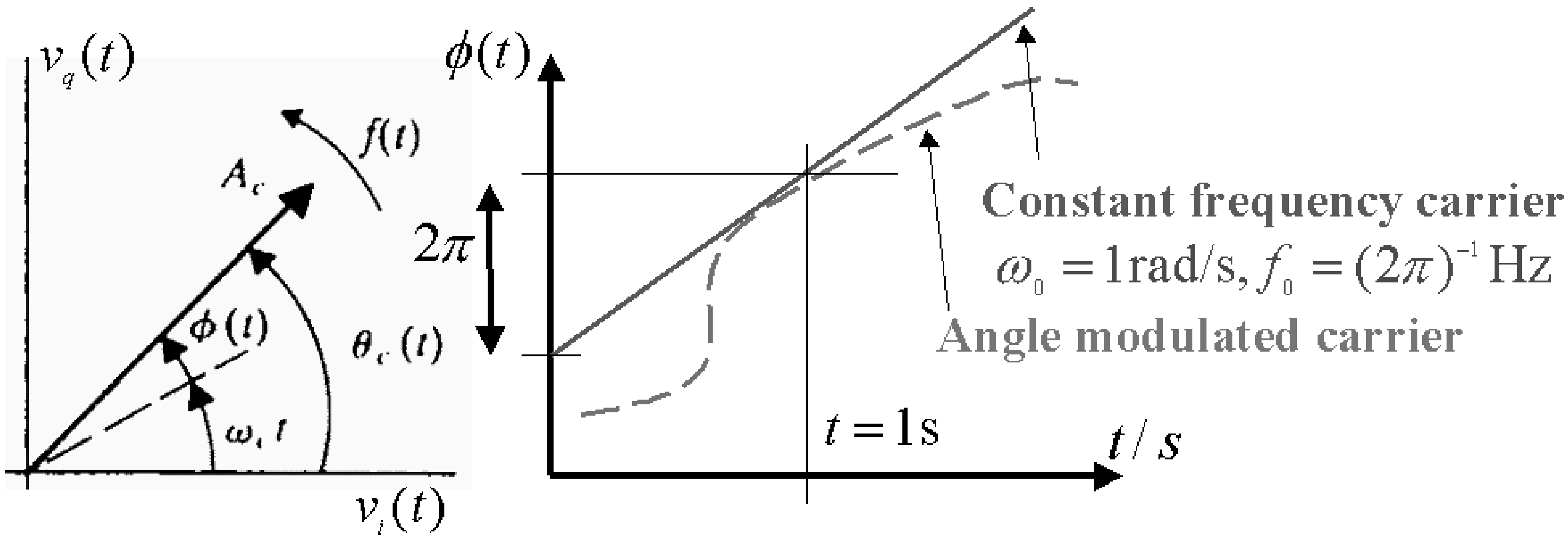
$$x_{PM}(t) = A_c \cos(\underbrace{\omega_c t + \underbrace{\phi(t)}_{\phi_\Delta x(t),\phi_\Delta \le \pi}}_{\theta_C(t)})$$

- Angular phasor has the instantaneous frequency (*phasor rate*)

$$\omega = 2\pi f(t)$$



Instantaneous frequency



- Angular frequency ω (rate) is the derivative of the phase (the same way as the velocity *v(t)* is the derivative of distance *s(t)*)
- For continuously changing frequency instantaneous frequency is defined by differential changes:

$$\omega(t) = \frac{d\phi(t)}{dt} \quad \phi(t) = \int_{-\infty}^{t} \omega(\alpha)d\alpha$$

Compare to linear motion: $v(t) = \frac{ds(t)}{dt}\left(\approx \frac{s_2(t) - s_1(t)}{t_2 - t_1}\right)$

Frequency modulation (FM)

- In frequency modulation carrier instantaneous frequency is linearly proportional to modulation frequency:

$$\omega = 2\pi f(t) = d\theta_C(t)/dt$$
$$= 2\pi[f_C + f_\Delta x(t)]$$

- Hence the FM waveform can be written as

$$x_C(t) = A_C \cos(\underbrace{\omega_C t + 2\pi f_\Delta \int_{t_0}^{t} x(\lambda)d\lambda}_{\theta_C(t)}), t \geq t_0$$

$$\phi(t) = \int_{-\infty}^{t} \omega(\alpha)d\alpha$$

integrate ←

- Note that for FM

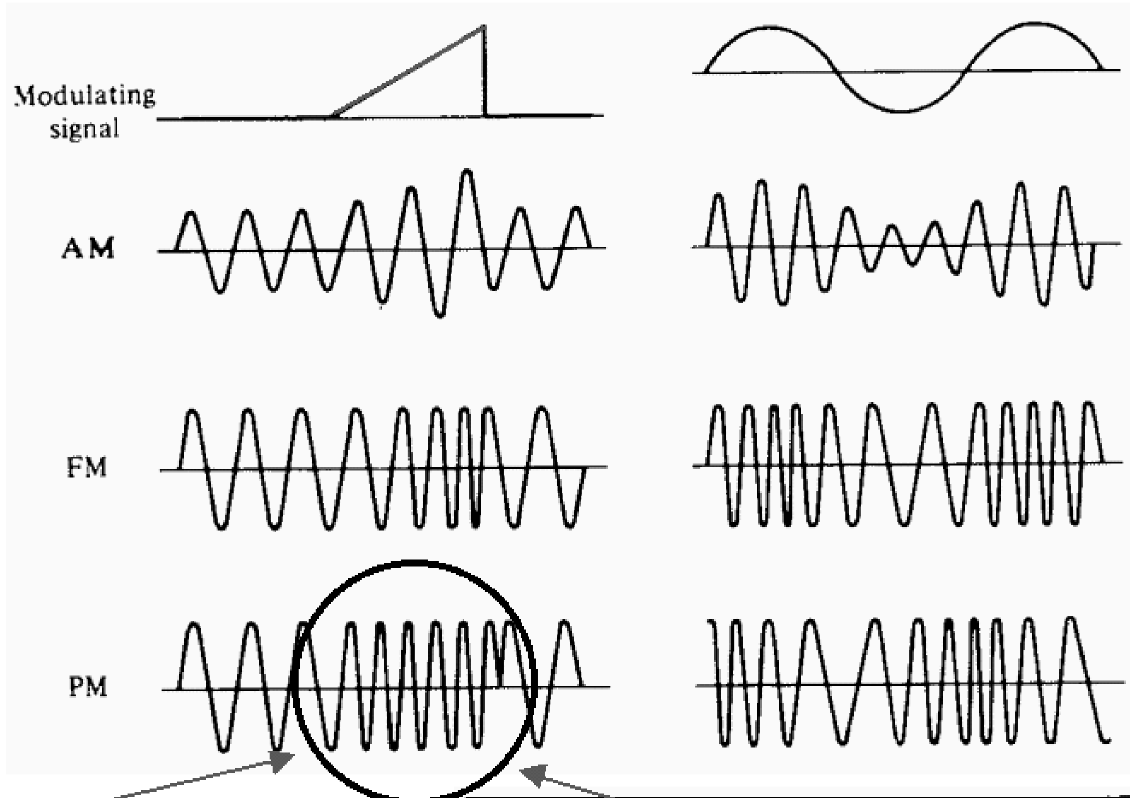
$$f(t) = f_C + f_\Delta x(t)$$

and for PM

$$\phi(t) = \phi_\Delta x(t)$$

	Instantaneous phase $\phi(t)$	Instantaneous frequency $f(t)$
PM	$\phi_\Delta x(t)$	$f_c + \frac{1}{2\pi}\phi_\Delta \dot{x}(t)$
FM	$2\pi f_\Delta \int^t x(\lambda)\,d\lambda$	$f_c + f_\Delta x(t)$

AM, FM and PM waveforms



Constant frequency at slope: follows the derivative of the modulation waveform

$$x_{PM}(t) = A_C \cos(\omega_C t + \phi_\Delta x(t))$$

$$x_{FM}(t) = A_C \cos(\omega_C t + 2\pi f_\Delta \int_t x(\lambda) d\lambda)$$

	Instantaneous phase $\phi(t)$	Instantaneous frequency $f(t)$
PM	$\phi_\Delta x(t)$	$f_c + \frac{1}{2\pi} \phi_\Delta \dot{x}(t)$
FM	$2\pi f_\Delta \int^t x(\lambda)\, d\lambda$	$f_c + f_\Delta x(t)$

Narrowband FM and PM (small modulation index, arbitrary modulation waveform)

- The CW presentation: $x_c(t) = A_c \cos[\omega_c t + \phi(t)]$
- The quadrature CW presentation:

$$x_c(t) = x_{ci}(t)\cos(\omega_c t) - x_{cq}(t)\sin(\omega_c t)$$

$$x_{ci}(t) = A_c \cos\phi(t) = A_c[1 - (1/2!)\phi^2(t) + \ldots]$$

$$x_{cq}(t) = A_c \sin\phi(t) = A_c[\phi(t) - (1/3!)\phi^3(t) + \ldots]$$

- The narrow band condition: $|\phi(t)| << 1$rad

$$x_{ci}(t) \approx A_c \quad x_{cq}(t) \approx A_c\phi(t)$$

- Hence the Fourier transform of $X_C(t)$ is in this case

$$\mathbb{F}\left[x_C(t)\right] \approx \mathbb{F}\left[A_C\cos(\omega_C t) - A_C\phi(t)\sin(\omega_C t)\right]$$

$$X_c(f) \approx \frac{1}{2}A_c\delta(f - f_c) + \frac{j}{2}A_c\Phi(f - f_c), f > 0$$

$$\mathbb{F}\left[\cos(2\pi f_0 t)\right] = \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

$$\mathbb{F}\left[\cos(2\pi f_0 t + \theta)x(t)\right] = \frac{1}{2}\left[X(f - f_0)\exp(j\theta) + jX(f + f_0)\exp(-j\theta)\right]$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

Narrow band FM and PM spectra

- Instantaneous phase in CW presentation:

$$x_c(t) = A_c \cos[\omega_c t + \phi(t)]$$

$$\phi_{PM}(t) = \phi_\Delta x(t)$$

$$\phi_{FM}(t) = 2\pi f_\Delta \int_{t_0}^{t} x(\lambda) d\lambda, t \geq t_0$$

- The small angle assumption produces compact spectral presentation for both FM and AM:

$$X_c(f) \approx \frac{1}{2} A_c \delta(f - f_c) + \frac{j}{2} A_c \Phi(f - f_c), f > 0$$

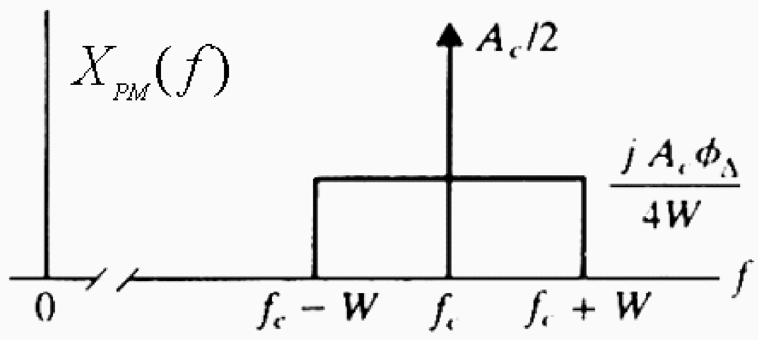
$$\Phi(f) = \mathbb{F}[\phi(t)]$$

$$= \begin{cases} \phi_\Delta X(f), \text{PM} \\ -jf_\Delta X(f) / f, \text{FM} \end{cases}$$

What does it mean to set this component to zero?

$$\int_{t_0}^{t} g(\tau) d\tau \Leftrightarrow \frac{G(\omega)}{j\omega} + \pi G(0)\delta(\omega)$$

Example



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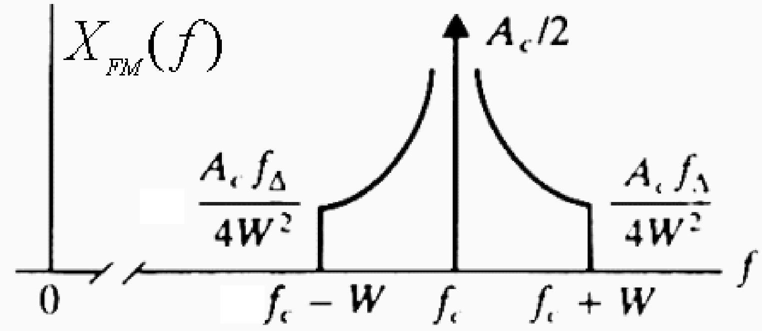


(b)

- Assume: $x(t) = \text{sinc}\,2Wt \Rightarrow X(f) = \frac{1}{2W}\Pi\left(\frac{f}{2W}\right)$

$$X_C(f) \approx \frac{1}{2}A_C\delta(f - f_C) + \frac{j}{2}A_C\Phi(f - f_C), f > 0$$

$$\Phi_{PM}(f) = F[\phi_{PM}(t)] = \phi_\Delta X(f) \qquad \Phi_{FM}(f) = F[\phi_{FM}(t)] = -jf_\Delta X(f)/f$$

$$X_{PM}(f) \approx \frac{1}{2}A_C\delta(f - f_C) + \frac{j}{4W}A_C\phi_\Delta\Pi\left(\frac{f - f_C}{2W}\right), f > 0$$

$$X_{FM}(f) \approx \frac{1}{2}A_C\delta(f - f_C) + \frac{f_\Delta}{4|f - f_C|W}A_C\Pi\left(\frac{f - f_C}{2W}\right), f > 0$$

Tone modulation with PM and FM: modulation index β

- Remember the FM and PM waveforms:

$$x_{PM}(t) = A_C \cos[\omega_C t + \underbrace{\phi_\Delta x(t)}_{\phi(t)}]$$

$$x_{FM}(t) = A_C \cos[\omega_C t + \underbrace{2\pi f_\Delta \int_t x(\lambda)d\lambda}_{\phi(t)}]$$

- Assume tone modulation

$$x(t) = \begin{cases} A_m \sin(\omega_m t), \text{PM} \\ A_m \cos(\omega_m t), \text{FM} \end{cases}$$

- Then

$$\phi(t) = \begin{cases} \phi_\Delta x(t) = \underbrace{\phi_\Delta A_m}_{\beta} \sin(\omega_m t), \text{PM} \\ 2\pi f_\Delta \int_t x(\lambda)d\lambda = \underbrace{(A_m f_\Delta / f_m)}_{\beta} \sin(\omega_m t), \text{FM} \end{cases}$$

FM and PM with tone modulation and arbitrary modulation index

- Time domain expression for FM and PM:

$$x_c(t) = A_c \cos[\omega_c t + \beta \sin(\omega_m t)]$$

$$\beta_{PM} = \phi_\Delta A_m$$
$$\beta_{FM} = A_m f_\Delta / f_m$$

- Remember: $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

- Therefore:

$$x_c(t) = A_c \cos(\beta \sin(\omega_m t))\cos(\omega_c t) - A_c \sin(\beta \sin(\omega_m t))\sin(\omega_c t)$$

$$\cos(\beta \sin(\omega_m t)) = J_o(\beta) + \sum_{n\,\text{even}}^{\infty} 2J_n(\beta)\cos(n\omega_m t)$$
$$\sin(\beta \sin(\omega_m t)) = \sum_{n\,\text{odd}}^{\infty} 2J_n(\beta)\sin(n\omega_m t)$$

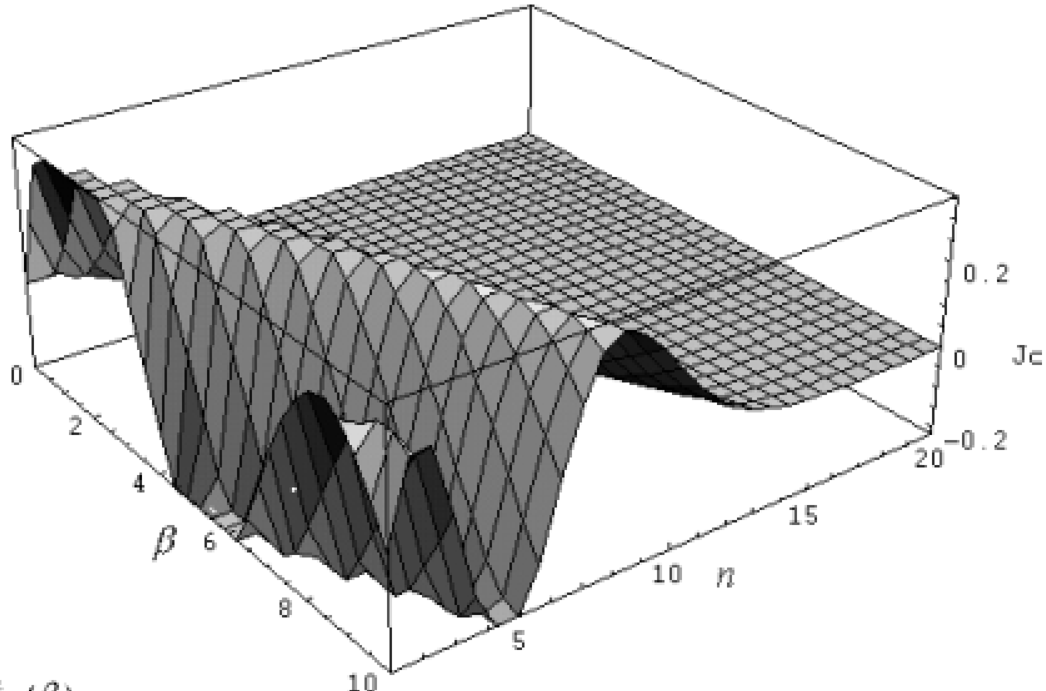
J_n is the first kind, order n Bessel function

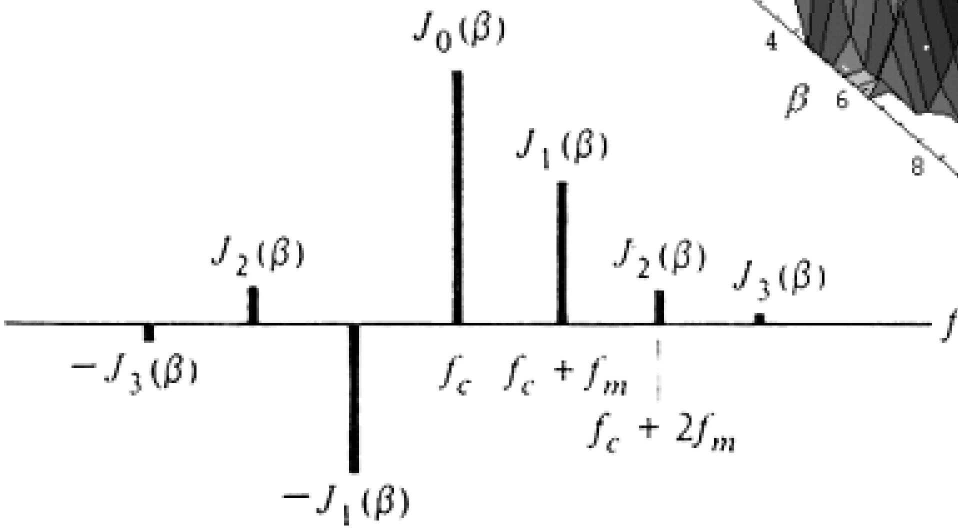
Wideband FM and PM spectra

- After simplifications we can write:

$$x_c(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\cos(\omega_c + n\omega_m)t$$

$$\beta = \begin{cases} \phi_\Delta A_m, \text{PM} \\ A_m f_\Delta / f_m, \text{FM} \end{cases}$$

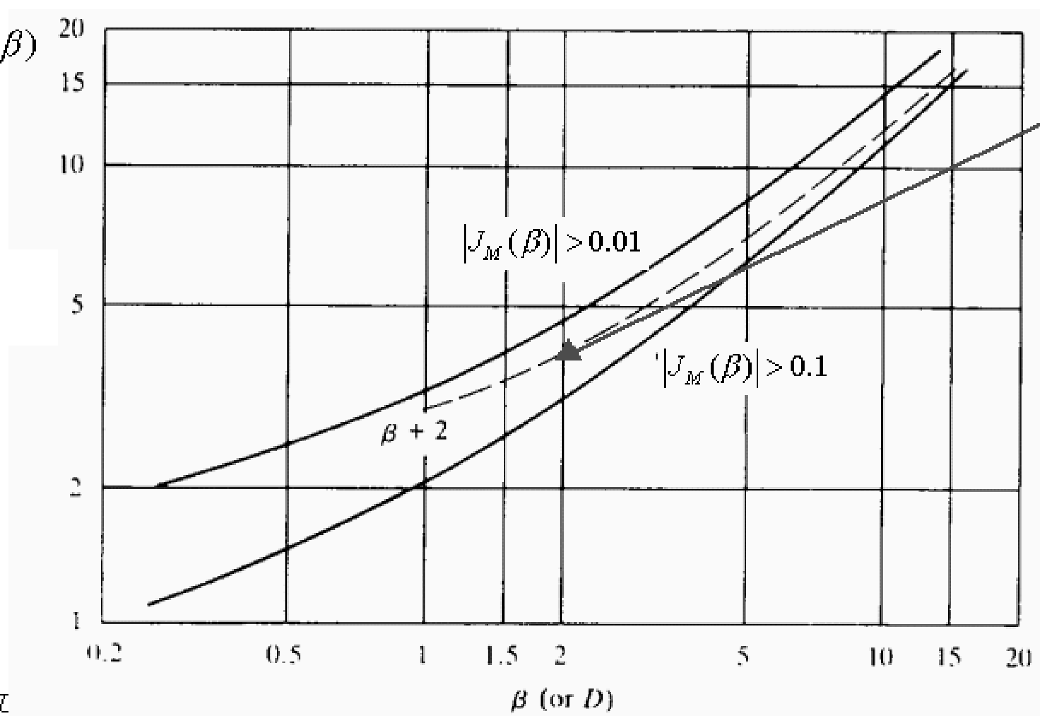




note: $J_{-n}(\beta) = (-1)^n J_n(\beta)$

Determination of transmission bandwidth

- The goal is to determine the number of significant sidebands
- Thus consider again how Bessel functions behave as the function of β, e.g. we consider $A_m \le 1, f_m \le W$
- Significant sidebands: $|J_n(\beta)| > \varepsilon$
- Minimum bandwidth includes 2 sidebands (why?): $B_{T,\min} = 2f_m$
- Generally: $B_T = 2M(\beta)f_m, M(\beta) \ge 1$



$M(\beta) \approx \beta + 2$

$$\beta = \begin{cases} \phi_\Delta A_m, \text{PM} \\ A_m f_\Delta / f_m, \text{FM} \end{cases}$$

Transmission bandwidth and deviation D

- Tone modulation is extrapolated into arbitrary modulating signal by defining deviation by

$$\beta = A_m f_\Delta / f_m \big|_{A_m=1, f_m=W} = f_\Delta / W \equiv D$$

- Therefore transmission BW is also a function of deviation

$$B_T = 2M(D)W$$

- For very large D and small D with

$$B_T \approx 2(D + \not{2}) f_m \big|_{D>>1, f_m=W}$$
$$\approx 2DW, D >> 1$$

$$B_T = 2M(D)W$$
$$\approx 2W, D << 1 \text{ (a single pair of sidebands)}$$

- that can be combined into

$$B_T = 2|D-1|W, D >> 1, \text{and } D << 1$$

$$\beta = \begin{cases} \phi_\Delta A_m, \text{PM} \\ A_m f_\Delta / f_m, \text{FM} \end{cases}$$

Example: Bandwidth of FM broadcasting

- Following commercial FM specifications

$$f_\Delta = 75\,\text{kHz}, W \approx 15\ \text{kHz}$$
$$\Rightarrow D = f_\Delta / W = 5$$
$$B_T = 2(D+2)W \approx 210\,\text{kHz}, (\text{D} > 2)$$

- High-quality FM radios RF bandwidth is about

$$B_T \geq 200\,\text{kHz}$$

- Note that

$$B_T = 2|D-1|W \approx 180\,\text{kHz},\ D >> 1$$

under estimates the bandwidth slightly

Generation of FM or PM by VCO

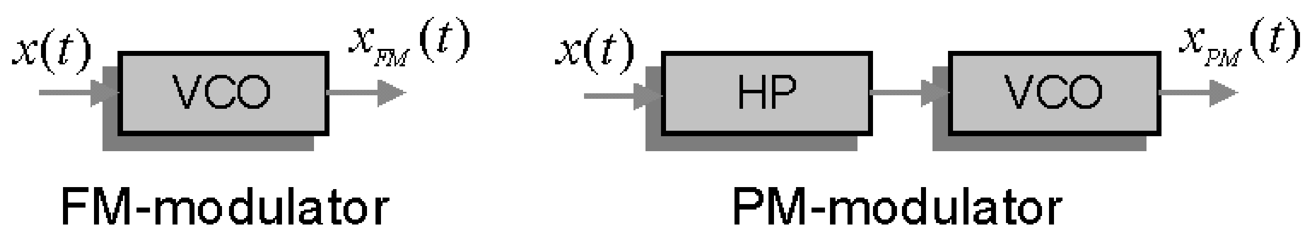
- Output signal of Voltage Controlled Oscillator (VCO) is

$$f_0(t) = f_c + K_D v_i(t)$$

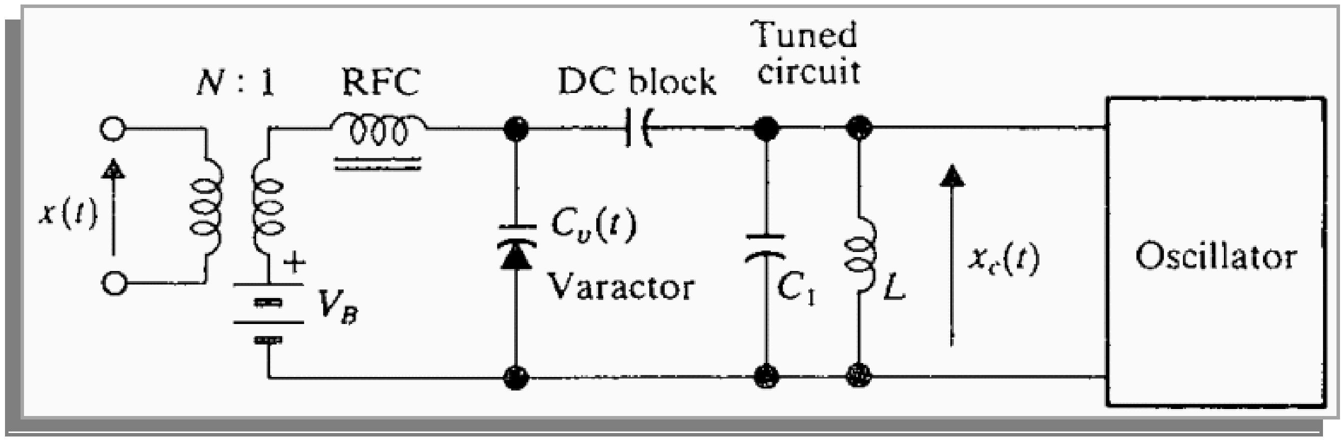
- This is precisely how instantaneous frequency of FM was defined: $f(t) = f_c + f_\Delta x(t)$
- VCO can be used to produce also PM :

$$f(t) = f_c + \frac{1}{2\pi}\phi_\Delta x'(t)$$

- Required differentiation can be realized by a high pass (HP) filter



Generating FM



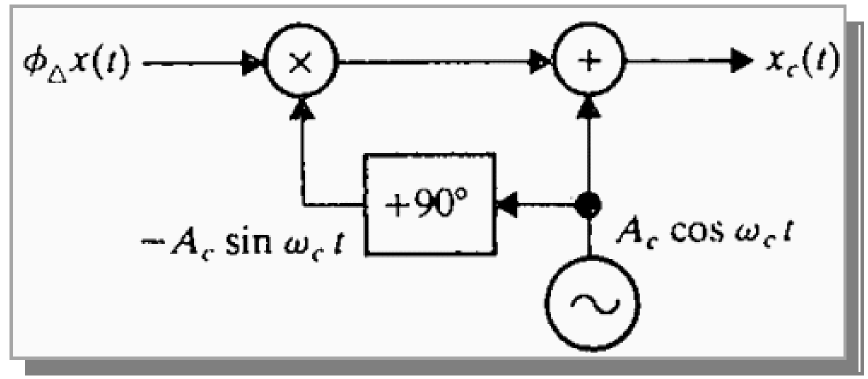
- A de-tuned resonant circuit oscillator
  - biased varactor diode capacitance directly proportional to *x(t)*
  - other parts:
    - input transformer
    - RF-choke
    - DC-block
- See the detailed analysis in lecture supplementary material

Narrow band mixer modulator

- Integrating the input signal to a phase modulator produces frequency modulation -> PM modulator is applicable for FM

$$x_{PM}(t) = A_C \cos(\omega_C t + \phi_\Delta x(t))$$
$$x_{FM}(t) = A_C \cos(\omega_C t + 2\pi f_\Delta \int_t x(\lambda) d\lambda)$$

- Also, PM can be produced by an FM modulator by differentiating its input
- Narrow band mixer modulator:



$$X_C(f) \approx \frac{1}{2} A_C \delta(f - f_C)$$
$$+ \frac{j}{2} A_C \phi_\Delta \Phi(f - f_C), f > 0$$

$$\phi_{PM}(t) = \phi_\Delta x(t), \phi_{FM}(t) = 2\pi f_\Delta \int_t x(\lambda) d\lambda$$

$$\Phi(f) = \mathbb{F}[\phi(t)] = \begin{cases} \phi_\Delta X(f), \text{PM} \\ -j f_\Delta X(f)/f, \text{FM} \end{cases}$$

Demodulation of FM

- FM demodulator examples:
  - FM-AM conversion followed by envelope detector
  - Phase-shift discriminator
  - Zero-crossing detection
  - PLL-detector
- FM-AM conversion is produced by a transfer function having magnitude distortion - example: derivative:

$$x_C(t) = A_C \cos(\omega_0 t + \phi(t))$$

$$\frac{dx_C(t)}{dt} = -A_C \sin[\omega_C t + \phi(t)](\omega_C + d\phi(t)/dt)$$

$$d\phi(t)/dt = 2\pi f(t)$$

$$= 2\pi[f_C + f_\Delta x(t)] \text{ FM}$$

Modulation moves to envelope

- Mathematica® example:

```
In[14]:= D[Cos[ωc t + Am Integrate[f[t], t]], t] // Expand

Out[14]= -f[t] Sin[(∫f[t] ⅆt) Am + t ωc] Am -
          Sin[(∫f[t] ⅆt) Am + t ωc] ωc
```

FM-AM conversion based PM detector

- Differentiation of the PM-wave produces FM-AM conversion:

In[5]:= **D[Cos[ω_C t + A_m f[t]], t] // Expand**

Out[5]= - Sin[f[t] A_m + t ω_C] ω_C -
Sin[f[t] A_m + t ω_C] A_m f′[t]

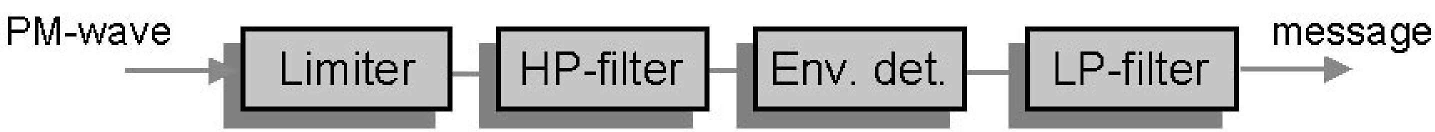
$$x_{PM}(t) = A_c \cos[\omega_c t + \underbrace{\phi_\Delta x(t)}_{\phi(t)}]$$

$$x_{FM}(t) = A_c \cos[\omega_c t + \underbrace{2\pi f_\Delta \int^t x(\lambda)d\lambda}_{\phi(t)}]$$

- Where after filtering the carrier, envelope detector yields

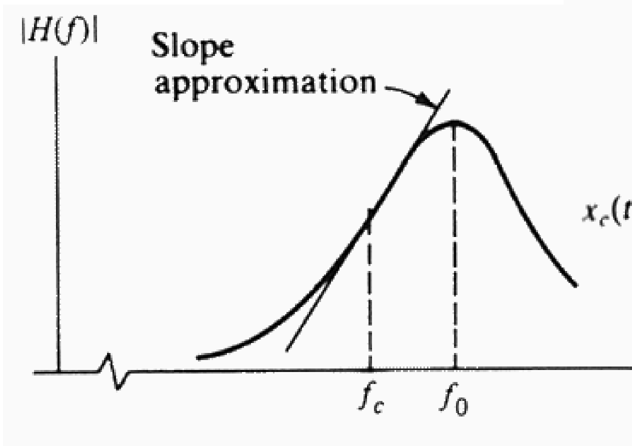
A_m f′[t]

whose integration (realized by an LP-filter) yields detected PM wave



- How one should select the 3 dB corners of the LP-filters in this application?
- See supplementary material for a proof that integration can be approximated by LP filter

FM slope detector and balanced discriminator are based on FM-AM conversion



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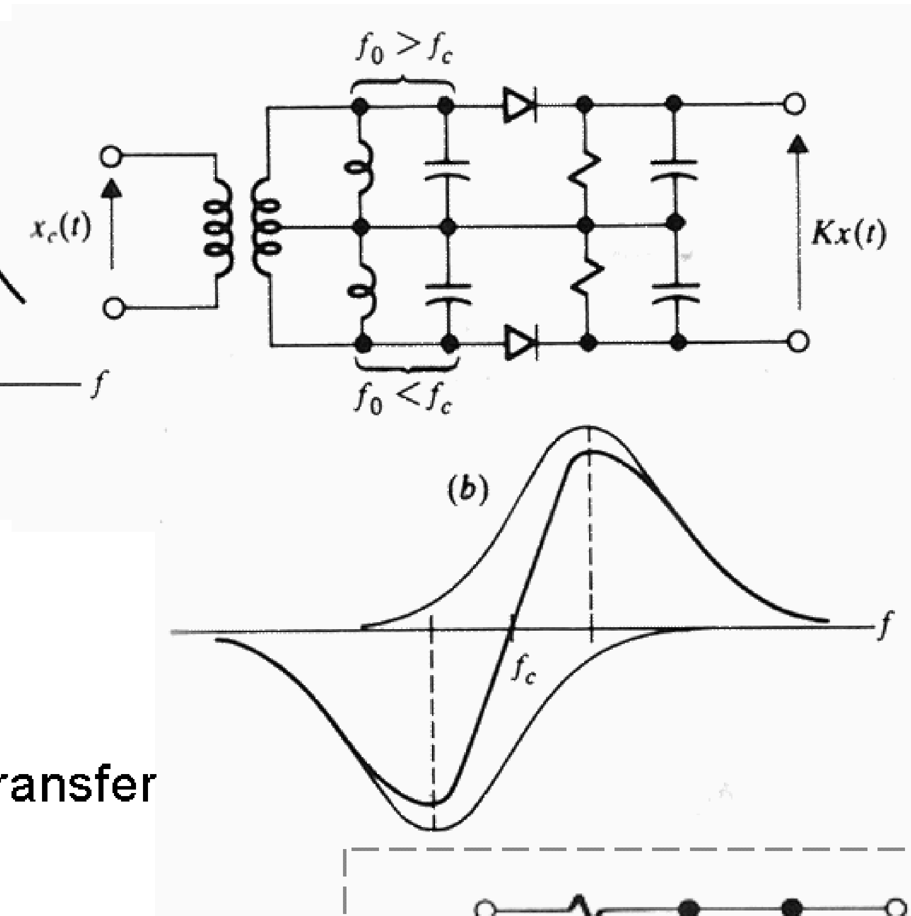
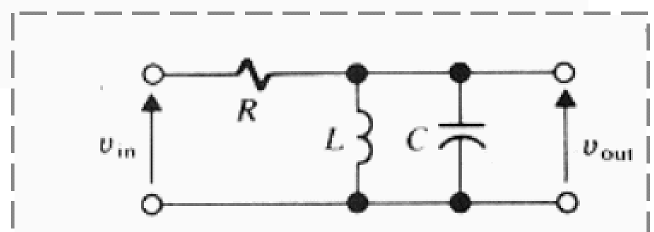


(b)

a) slope detector realized
by tank-circuit

b) dual-slope detector and transfer
characteristics

c) tank circuit



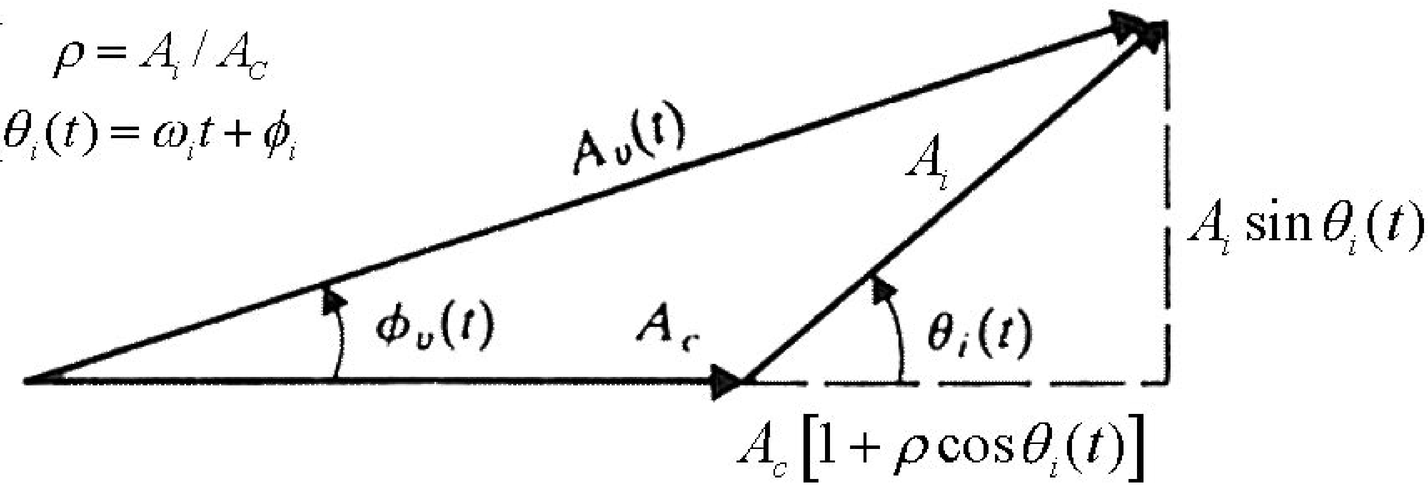
(c)

Additive interference in unmodulated carrier

- Consider a general tone interference signal

$$v(t) = \underbrace{A_C \cos(\omega_C t)}_{\text{carrier}} + \underbrace{A_i \cos\left[(\omega_C + \omega_i)t + \phi_i\right]}_{\text{interference}}$$

$$\begin{cases} \rho = A_i / A_C \\ \theta_i(t) = \omega_i t + \phi_i \end{cases}$$



- interference produces both AM and FM:

$$A_v(t) = A_C \sqrt{1 + \rho^2 + 2\rho\cos\theta_i(t)}$$

$$\phi_v(t) = \arctan\frac{\rho\sin\theta_i(t)}{1 + \rho\cos\theta_i(t)}$$

Additive interference and demodulators

- Further simplification under weak interference: $A_i << A_c, \rho << 1$

$$A_v(t) = A_C\sqrt{1 + \underbrace{\rho^2}_{\approx \rho^2\cos^2\theta_i(t)} + 2\rho\cos\theta_i(t)} \approx A_C\left[1 + \rho\cos\theta_i(t)\right]$$

$$\phi_v(t) = \arctan\frac{\rho\sin\theta_i(t)}{1+\rho\cos\theta_i(t)} \approx \arctan\left[\rho\sin\theta_i(t)\right] \approx \rho\sin\theta_i(t)$$

- Demodulation functions:

$$y_D(t) \approx \begin{cases} K_{D,AM}A_v(t), \text{AM} \\ K_{D,PM}\phi(t), \text{PM} \\ K_{D,FM}d\phi(t)/dt, \text{FM} \end{cases}$$

- And therefore

$$y_D(t) \approx \begin{cases} K_{D,AM}A_C\left[1+\rho\cos\theta_i(t)\right], \text{AM} \\ K_{D,PM}\rho\sin\theta_i(t), \text{PM} \\ K_{D1,FM}\rho f_i\cos\theta_i(t), \text{FM} \end{cases}$$

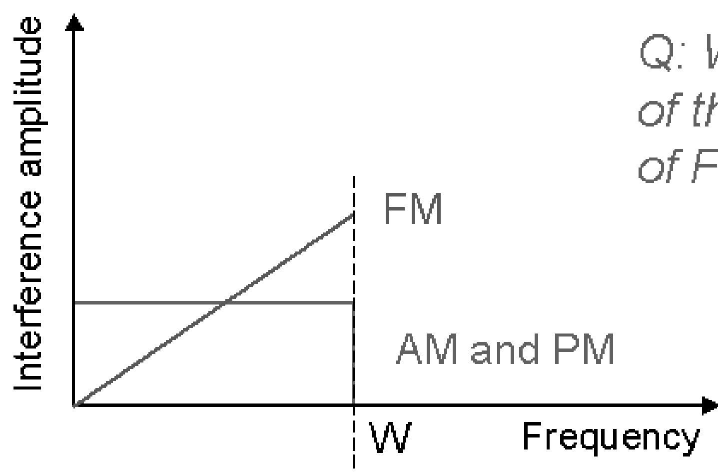
$$d\left[\rho\sin\theta_i(t)\right]/dt$$
$$= d\left[\rho\sin(\omega_i t + \phi_i)\right]/dt$$
$$= \rho\omega_i\cos(\omega_i t + \phi_i)$$

Implications for demodulator design

$$y_D(t) \approx \begin{cases} K_{D,AM} A_C\left[1+\rho\cos\theta_i(t)\right], \mathrm{AM} \\ K_{D,PM}\,\rho\sin\theta_i(t), \mathrm{PM} \\ K_{D,FM}\,\rho f_i\cos\theta_i(t), \mathrm{FM} \end{cases}$$

$$\rho = A_i / A_C$$
$$\theta_i(t) = \omega_i t + \phi_i$$

- In AM and PM a tone interference produces a tone to reception whose amplitude is comparable to ρ and position comparable to $\theta_i(t) = \omega_i t + \phi_i$
- Interference in FM is more severe the more remote the interfering tone is from the carrier (but still at the reception band *W*):



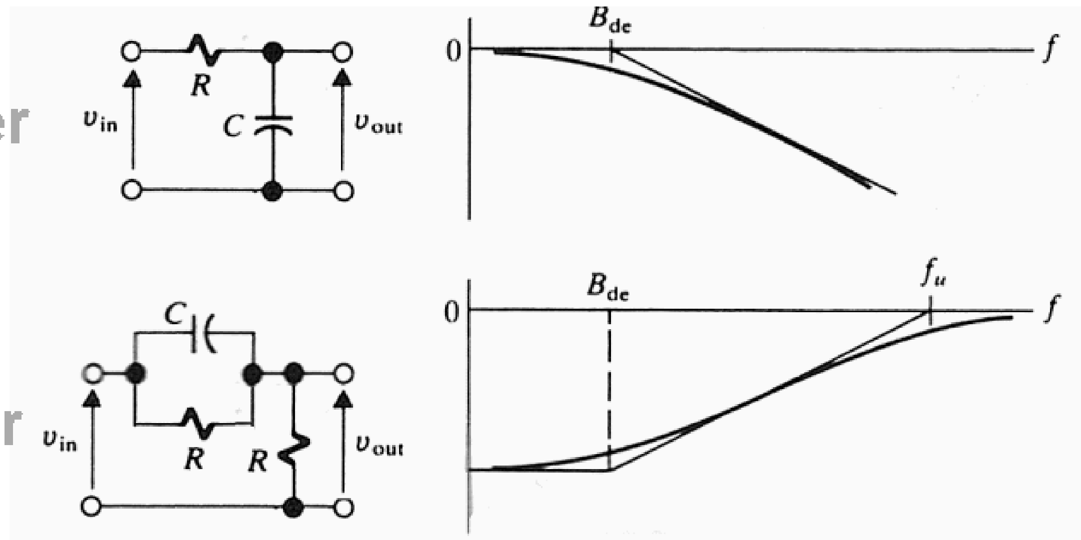
Q: What are the implications of this to noise sensitivity of FM bandwidth determination

FM preemphases and deemphases filters

- FM related noise emphases can be suppressed by *pre-distortion* and post detection filters (preemphases and deemphases filters):

receiver filter

transmitter filter



$H_{de}(f)$

$H_{pe}(f)$

Q: What would happen if the filters would be reversed? (TX filter in receiver & vice versa)

$$H_{de}(f) = \left[1 + j(f / B_{de})\right]^{-1} \approx \begin{cases} 1, |f| << B_{de} \\ B_{de} / (jf), |f| >> B_{de} \end{cases}$$

LPF

$$H_{pe}(f) = \left[1 + j(f / B_{de})\right] \approx \begin{cases} 1, |f| << B_{de} \\ j(f / B_{de}), |f| >> B_{de} \end{cases}$$

HPF