S-72.245 Transmission Methods in Telecommunication Systems (4 cr)

Noise in analog carrier wave (CW) modulation systems

Noise in analog CW modulation systems

- Understanding noise
  - Lowpass presentation of bandpass noise and its conversion to baseband noise
  - Noise statistics of quadrature presentation in rectangular and polar coordinates
- Modeling detectors for linear and exponential modulation
- Analysis of post-detection SNR
  - Synchronous detector
  - PM-detector
  - FM-detector

Noise in carrier wave modulation systems: basic definitions

- Objectives: Define bandpass noise and use it to analyze post detection SNR of analog CW systems
- Assume signal is ergodic, e.g., all ensemble averages E[] equal the corresponding time averages <>. Then, for instance

$$< v(t) >= \mathrm{E}[v(t)] \quad \text{average value}$$

$$< v^2(t) >= \mathrm{E}[v^2(t)] \quad \text{average power}$$

$$< v(t)v(t-\tau) >= \mathrm{E}[v(t)v(t-\tau)] \quad \text{autocorrelation}$$

where the time average is defined by

$$< v_i(t) >= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} v_i(t)dt$$

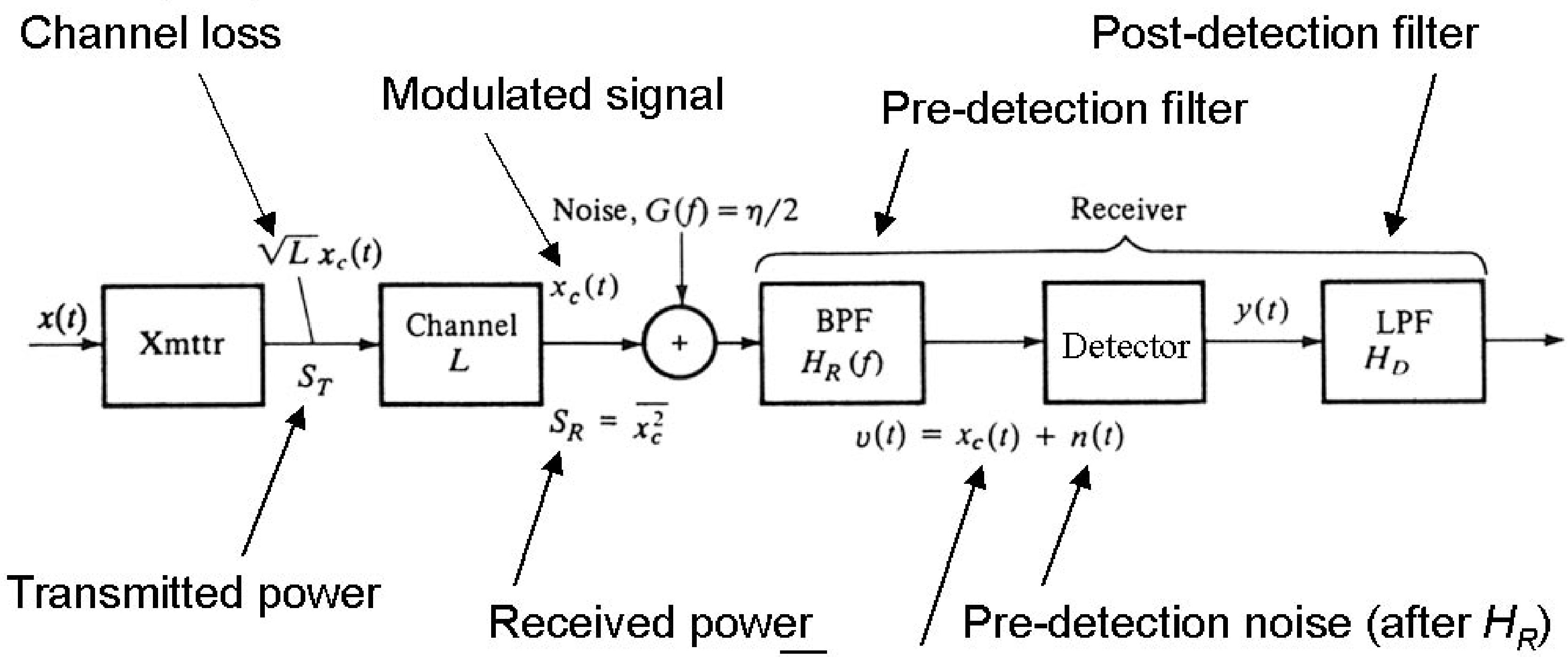
or

$$< v_i(t) >= \frac{1}{T} \int_{-T/2}^{T/2} v_i(t)dt \quad \text{(for a known period)}$$

The system model

- We consider normalized ergodic analog message whose amplitude and power are normalized

$$|x(t)| \le 1, \qquad S_x = \overline{x^2} = \langle x^2(t) \rangle, S_x \le 1$$



$$S_R = S_T / L = \overline{x_c^2}$$

Detection models

- Pre-detection signal *v(t)* is presented in quadrature-carrier form:

$$v(t) = A_v(t)\cos[\omega_c t + \phi_v(t)]$$
$$= v_i(t)\cos(\omega_c t) - v_q(t)\cos(\omega_c t)$$

$$\begin{cases} v_i(t) = A_v(t)\cos\phi(t) \\ v_q(t) = A_v(t)\sin\phi(t) \end{cases}$$

- Detection models:

$$y(t) = \begin{cases} v_i(t) & \text{Synchronous detector} \\ A_v(t) - \overline{A_v} & \text{Envelope detector} \\ \phi_v(t) & \text{Phase detector} \\ d\phi_v(t)/dt & \text{Frequency detector} \end{cases}$$

(Remember that FM was defined by:

$$x_C(t) = A_C \cos[\omega_C t + 2\pi f_\Delta \int_{t_0}^{t} x(\lambda)d\lambda], t \geq t_0)$$

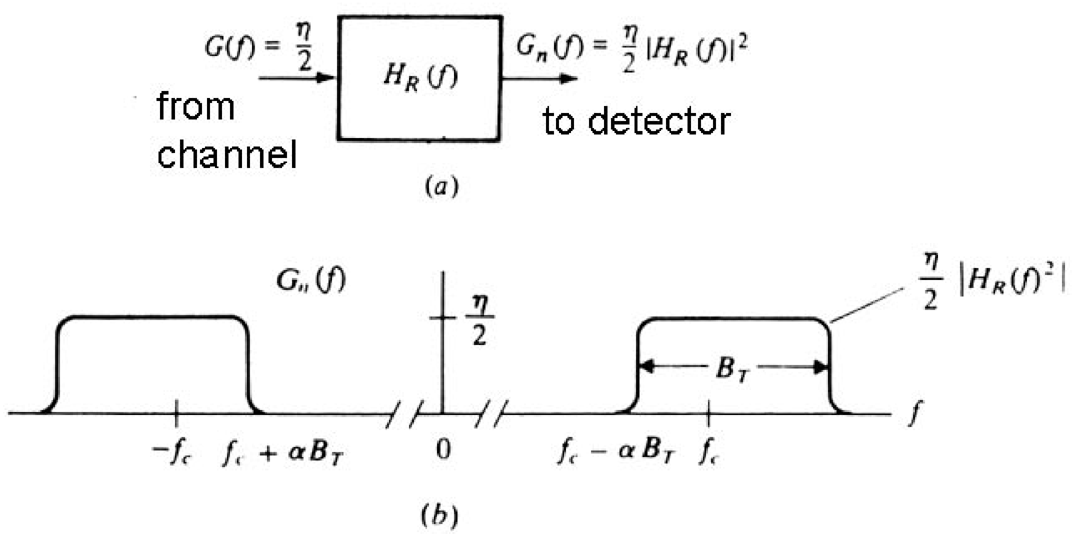
Pre-detection noise in bandpass channel

- Signal and noise are statistically independent and therefore their power can be added to form the total pre-detection power:

$$\overline{v^2} = \overline{x_c^2} + \overline{n^2} = S_R + N_R$$

- The pre-detection (bandpass) noise power is filtered from the channel noise:

$$N_R = \int_{-\infty}^{\infty} (\eta/2) \left| H_R(f) \right|^2 df = 2\int_0^{B_T} (\eta/2) df = \eta B_T$$



Pre-detection SNR

- Pre-detection signal-to-noise ratio for **bandpass** channels is defined by

$$S_R / N_R = S_R / (\eta B_T)$$

- Note that above B_T is the transmission bandwidth passing channel noise power to the detector

$$N_R = \eta B_T = (\eta / 2)\int_{-\infty}^{\infty} \left|H_R(f)\right|^2 df$$

- For comparison, we can write the received signal-to-noise in terms of **baseband system** (BW = W) SNR defined by

$$\gamma = S_R / (\eta W)$$

and therefore also

$$S_R / N_R = S_R / (\eta B_T) = (S_R / \eta W)(W / B_T) = \gamma W / B_T$$

- Note that always (limiting case is the SSB with $B_T = W$)

$$B_T \geq W \quad \Rightarrow \boxed{S_R / N_R \leq \gamma}$$

(We will see, however, that post detection SNR can be much larger than γ !)

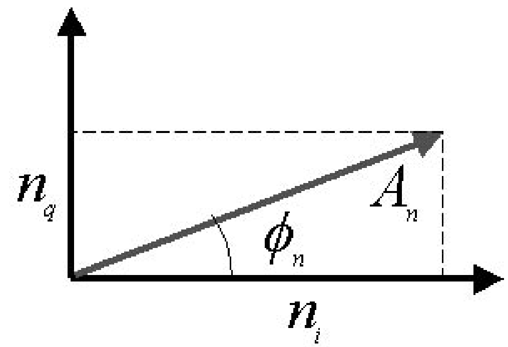
Bandpass noise: $n(t) = A_n(t)\cos[\omega_c t + \phi_n(t)]$

- We assume stationary, zero mean Gaussian noise process for which

$$\overline{n} = 0, \overline{n^2} = \sigma_n^2 = N_R$$

- Bandpass noise in terms of lowpass equivalent signals

$$n(t) = n_i(t)\cos(\omega_c t) - n_q(t)\sin(\omega_c t)$$



- The in-phase and quadrature components are independent and hence

$$\overline{n_i(t)n_q(t)} = 0$$

- Their average is zero $\overline{n_i} = \overline{n_q} = 0$ and their average power is the same:

$$\overline{n_i^2} = \overline{n_q^2} = \overline{n^2} = N_R$$

Bandpass noise has Rayleigh distributed envelope and evenly distributed phase

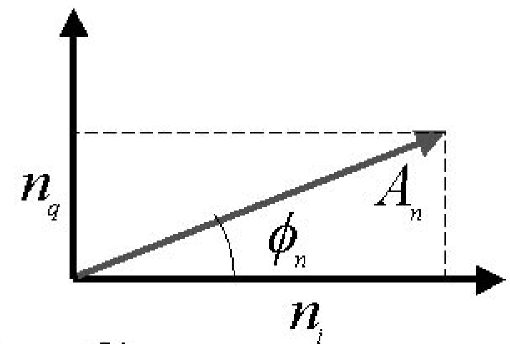
- I-Q components of the bandpass noise can be presented in envelope - phase format:

$$\overline{A_n^2} = \overline{n_i^2} + \overline{n_q^2}, \phi_n = \arctan \frac{n_q}{n_i}$$

Two independent r.v.:s - sum of their variances equals variance of the envelope

- The PDF of envelope is Rayleigh distributed defined by

$$p_{A_n}(A_n) = \frac{A_n^2}{N_R} \exp\left(-\frac{A_n^2}{2N_R}\right) u(A_n)$$



- Therefore mean and variance for the bandpass noise are (integrate from above, how?)

$$\overline{A_n} = \sqrt{\pi N_R / 2}, \ \overline{A_n^2} = 2N_R$$

$$\Rightarrow N_R = \overline{A_n^2}/2 = (\overline{n_i^2} + \overline{n_q^2})/2 \Big|_{\overline{n_i^2}=\overline{n_q^2}=\overline{n^2}} = \overline{n^2}$$

Post detection noise in synchronous detection

- **Signal component** of synchronous detector:

$$v_{DSB}(t) = x(t)\exp(j\omega_c t)$$

$$v_{DSB}(t) = A_c x(t)\cos(\omega_c t) + jA_c x(t)\sin(\omega_c t)$$

received DSB signal

$$v_{DSB}(t)\cos(\omega_c t) = \frac{A_c x(t)}{2}$$

← detected message

$$+\frac{A_c x(t)}{2}\cos(2\omega_c t) + j\frac{A_c x(t)}{2}\sin(2\omega_c t)$$

- **Noise component** of synchronous detector:

$$n(t)\cos(\omega_c t) = \cos(\omega_c t)\left[n_i(t)\cos(\omega_c t) + n_q(t)\sin(\omega_c t)\right]$$

$$= n_i(t)\left[1 + \cos(2\omega_c t)\right]/2 + n_q(t)\sin(2\omega_c t)/2$$

- Detector extracts *i*-components and removes double frequency components

$$y_D(t) = A_c x(t) + n_i(t)$$

$$\cos(x)\cos(x) = 1/2 + \cos(2x)/2$$
$$\cos(x)\sin(x) = \sin(2x)/2$$

Post-detection SNR for DSB

- Obtain signal and noise power after detection from:

$$y_D(t) = A_c x(t) + n_i(t)$$

where average noise and signal power are

$$N_D = \langle n_i^2(t) \rangle, \; S_D = \langle A_c^2 x^2(t) \rangle = A_c^2 S_x$$

Received average signal power is

$$S_R = \langle A_c^2 \underbrace{x^2(t)}_{S_x} \underbrace{\cos^2(\omega_0 t)}_{1/2} \rangle = A_c^2 S_x / 2 \Rightarrow S_x = 2S_R / A_c^2$$

and therefore SNR after DSB detector is

$$S_D / N_D = A_c^2 S_x / N_D$$

$$= \frac{\cancel{A_c^2}\, 2S_R}{\cancel{A_c^2}\, N_D} = \frac{2S_R}{N_D} = \left.\frac{2S_R}{\eta B_T}\right|_{B_T=2W \text{ (DSB)}} = \frac{S_R}{\eta W} = \gamma \text{ (DSB)}$$

Comparing SNR for DSB and AM

- It can be show, that for AM the post detection SNR is

$$S_D / N_D \big|_{AM} = \gamma \frac{S_x}{1+S_x}$$

- Comparison of this to the SNR of DSB can done by noting that in practice

$$S_x = 0.5 \text{ tone modulation}$$

$$S_x \approx 0.1 \text{ speech signal}$$

- Hence AM performs usually much worse than DSB
- It can be shown that for SSB performance is the same as for DSB, e.g.

$$S_D / N_D \big|_{USB,DSB} = \frac{S_R}{\eta W} = \gamma$$

Exponential modulation and channel noise

- Both PM and FM have constant envelopes so the received power is constant

$$x_c(t) = A_c \cos[\omega_c t + \phi(t)]$$

$$S_R = \overline{x_c^2} = A_c^2 / 2$$

- Received SNR is $\dfrac{S_R}{N_R} = \dfrac{A_c^2}{2N_R} = \dfrac{A_c^2}{2\eta B_T}$ yielding for wideband FM

$$\frac{S_R}{N_R} = \frac{S_R}{\eta B_T} = \frac{S_R}{\eta 2DW} = \frac{\gamma}{2D}$$

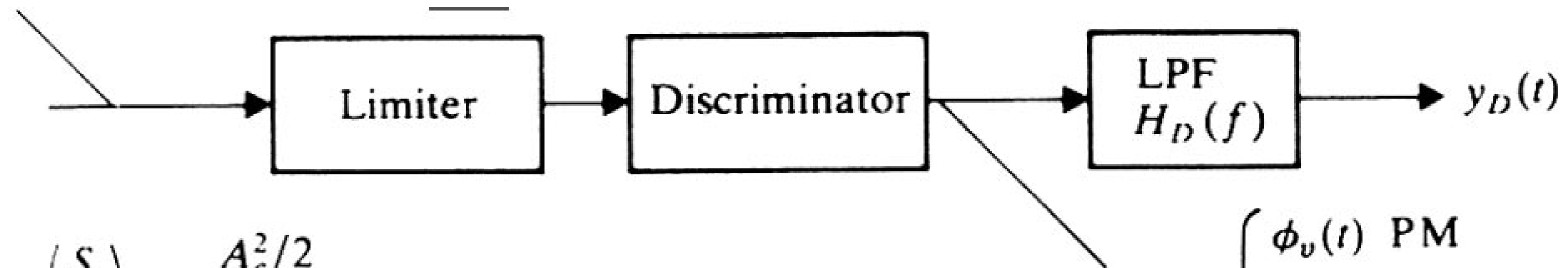
where for wideband modulation

$$B_T \approx 2(\beta + 2) f_m \big|_{\beta >> 1,\, f_m = W} \approx 2\beta W$$

$$= 2DW$$

Detection of exponential modulation assuming small noise power

$$v(t) = x_c(t) + n(t) = \underline{A_v(t)} \cos[\omega_c t + \phi_v(t)]$$



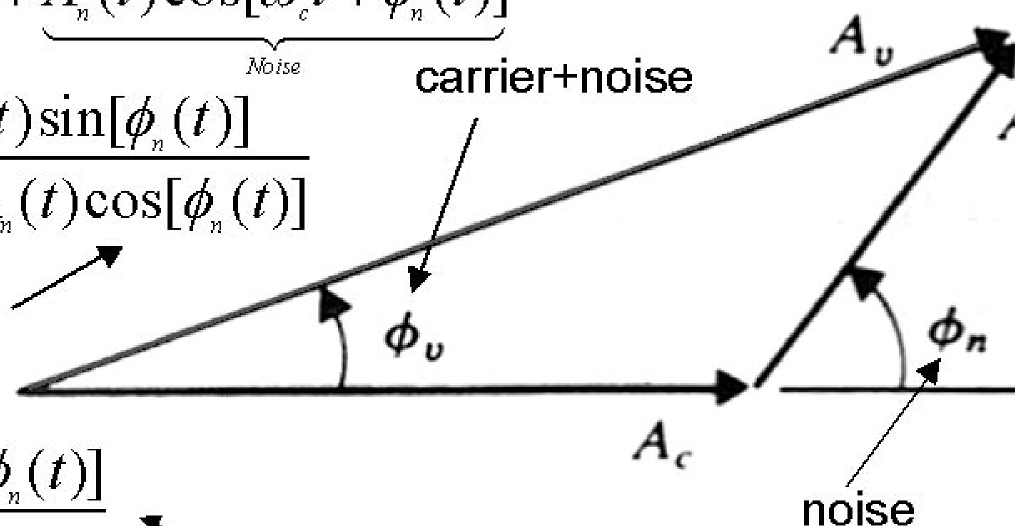
$$\left(\frac{S}{N}\right)_R = \frac{A_c^2/2}{\eta B_T}$$

$$y(t) = \begin{cases} \phi_v(t) & \text{PM} \\ \dfrac{1}{2\pi}\dot{\phi}_v(t) & \text{FM} \end{cases}$$

$$v(t) = \underbrace{A_c \cos[\omega_c t + \phi(t)]}_{Signal} + \underbrace{A_n(t)\cos[\omega_c t + \phi_n(t)]}_{Noise}$$

$$\phi_v(t) = \arctan\frac{A_n(t)\sin[\phi_n(t)]}{A_c + A_n(t)\cos[\phi_n(t)]}$$

small compared to A_c



$$\phi_v(t) \approx \frac{A_n(t)\sin[\phi_n(t)]}{A_c}$$

Detected noise component

Post-detection noise spectra for PM

$$\phi_v(t) \approx \frac{A_n(t)\sin\phi_n(t)}{A_c} = \frac{n_q(t)}{\sqrt{2S_R}}$$

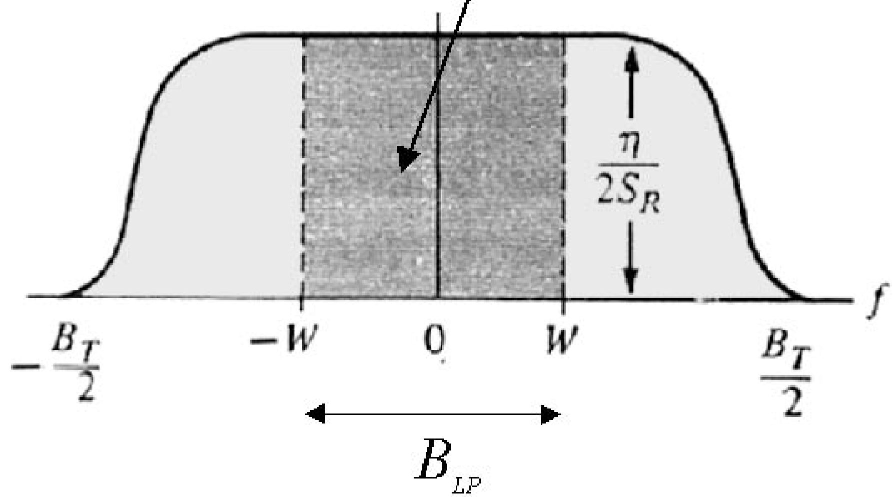
$$\begin{cases} S_R = A_c^2/2 \\ n_q = A_n \sin\phi_n \\ \overline{n_i^2} = \overline{n_q^2} = N_R = \eta B_{LP} \end{cases}$$

Note that after detection signal bandwidth is *W* and thus a post detection filter is required to remove out-of-band channel noise

- The channel noise is bandpass noise filtered at the transmission bandwidth and therefore the respective post-detection noise power spectral density $G_{PM}(f)$ and the total noise power N_D are

$$G_{PM}(f) = \left(\frac{n_q(t)}{\sqrt{2S_R}}\right)^2 \Pi\left(\frac{f}{B_T}\right) = \frac{\eta}{2S_R}\Pi\left(\frac{f}{B_T}\right)$$

$$N_D = \int_{-W}^{W} G_{PM}(f)df = \eta W / S_R$$



Post-detection SNR for FM

- Recall the definition of FM-signal
$$x_c(t) = A_C \cos\left[\omega_C t + \phi_v(t)\right], \phi_v(t) = 2\pi f_\Delta \int_{t_0}^{t} x(\lambda)d\lambda$$
- Frequency discriminator (detector) differentiates the instantaneous phase to cancel out the inherent integration in phase. Now $\omega(t) = 2\pi x(t) = d\phi_v(t)/dt \Rightarrow$

$$x(t) = \frac{d\phi_v(t)}{2\pi dt} = \frac{1}{2\pi}\left\{\underbrace{\frac{d\phi_S(t)}{dt}}_{Signal} + \underbrace{\frac{d\phi_N(t)}{dt}}_{Noise}\right\} = \frac{1}{2\pi}\left\{\frac{d\phi_S(t)}{dt} + \frac{d\left[n_q(t)\right]}{\sqrt{2S_R}dt}\right\}$$

- Inspection in frequency domain (In order to find the respective PSDs) yields after detector
$$X(f) = j2\pi f \Phi_v(f)$$
and the signal PSD is
$$G_{FM}(f) = \left|\hat{X}(f)\right|^2 = (2\pi f)^2 \left|\Phi_v(f)\right|^2 / (2\pi)^2 = f^2 \left|\Phi_v(f)\right|^2$$

$$\phi_v(t) \approx \frac{A_n(t)\sin\phi_n(t)}{A_c} = \frac{n_q(t)}{\sqrt{2S_R}} \qquad \frac{d^n x(t)}{dt^n} \leftrightarrow (j2\pi f)^n X(f)$$

Post-detection SNR in FM (cont.)

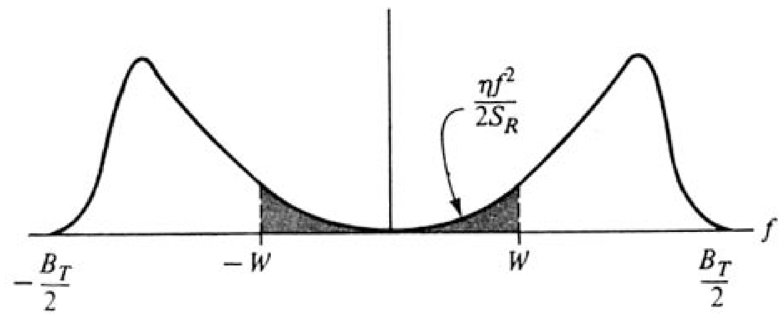
- Therefore, the post-detection noise PSD can be written as

$$G_{FM}(f) = f^2 \left|\Phi_v(f)\right|^2 \text{ with}$$

$$\Phi_v(f) = \frac{N_q(f)}{\sqrt{2S_R}}$$

and now the PSD for FM post detection noise is

$$G_{FM}(f) = \frac{\eta f^2}{2S_R} \Pi\left(\frac{f}{B_T}\right)$$



and the respective total noise power is

$$N_D = \int_{-W}^{W} G_{FM}(f)df = \eta W^3 / (3S_R)$$

 Helsinki University of Technology, Communications Laboratory, Timo O. Korhonen

Destination S/N for PM and FM

- For PM we have

$$S_D / N_D = \frac{\phi_\Delta^2 S_x}{\eta W / S_R} = \phi_\Delta^2 S_x \gamma, \text{where } \phi_\Delta^2 S_x \le \pi^2$$

- For FM we have

$$S_D / N_D = \frac{f_\Delta^2 S_x}{\eta W^3 / (3S_R)}$$

$$= 3\underbrace{\left(\frac{f_\Delta}{W}\right)}_{D}^2 S_x \underbrace{\frac{S_R}{\eta W}}_{\gamma} = \underline{3D^2 S_x \gamma}$$

$$N_{D(PM)} = \int_{-W}^{W} \frac{\eta}{2S_R} df = \frac{\eta W}{S_R}$$

$$N_{D(FM)} = \int_{-W}^{W} \frac{\eta f^2}{2S_R} df = \frac{\eta W^3}{3S_R}$$

Note that S_D/N_D can be increased just by increasing deviation!

- Under wideband condition $D >> 1$ and

$$B_T \approx 2(\beta + 2) f_m \Big|_{\beta >> 1, f_m = W} \approx 2\beta W$$

$$= 2DW \Rightarrow D = B_T / (2W)$$

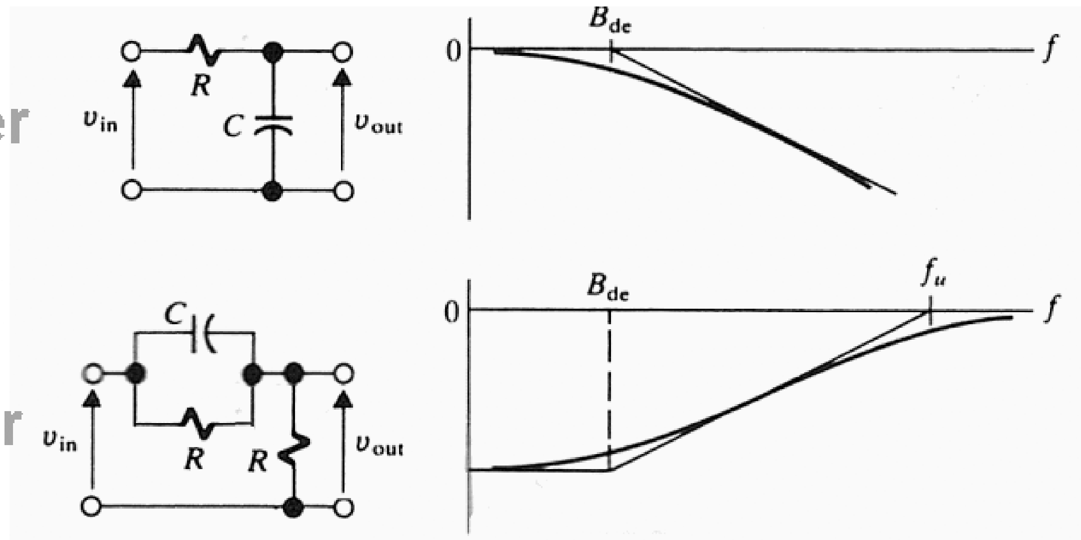
$$S_D / N_D = \underline{\frac{3}{4}\left(\frac{B_T}{W}\right)^2 S_x \gamma}$$

FM preemphases and deemphases filters

- FM related noise emphases can be suppressed by *pre-distortion* and post detection filters (preemphases and deemphases filters):

receiver filter

transmitter filter



$H_{de}(f)$

$H_{pe}(f)$

Q: What would happen if the filters would be reversed? (TX filter in receiver & vice versa)

$$H_{de}(f) = [1 + j(f / B_{de})]^{-1} \approx \begin{cases} 1, |f| << B_{de} \\ B_{de} / (jf), |f| >> B_{de} \end{cases}$$ LPF

$$H_{pe}(f) = [1 + j(f / B_{de})] \approx \begin{cases} 1, |f| << B_{de} \\ j(f / B_{de}), |f| >> B_{de} \end{cases}$$ HPF

FM post-detection S/N with deemphases

- Deemphases filter (that is a lowpass filter connected after detector) can suppress noise further. FM post-detection noise PSD and total noise power without deemphases:

$$G_{FM}(f) = \frac{\eta f^2}{2S_R}\Pi\left(\frac{f}{B_T}\right) \quad N_D = \int_{-W}^{W} G_{FM}(f)df = \frac{\eta W^3}{3S_R}$$

- With deemphases filter (for simplification assume W/B_{de}>>1):

$$N_D = \int_{-W}^{W} G_{FM}(f)\left|H_{de}(f)\right|^2 df = \frac{\eta B_{de}^3}{S_R}\left[\frac{W}{B_{de}} - \underbrace{\arctan\frac{W}{B_{de}}}_{<<W/B_{de}}\right] \approx \eta B_{de}^2 W / S_R$$

where

$$\left|H_{de}(f)\right|^2 = \frac{1}{1+(f/B_{de})^2}$$

$$S_D / N_D = \frac{f_\Delta^2 S_x}{\eta B_{de}^2 W / S_R} = \frac{S_x S_R}{\eta W}\left(\frac{f_\Delta}{B_{de}}\right)^2 = \underline{S_x \gamma\left(\frac{f_\Delta}{B_{de}}\right)^2}$$

Example

- FM radio

$$f_\Delta = 75\text{ kHz}, W = 15\text{ kHz}, D = 5, S_x = 1/2, B_{de} = 2.1\text{kHz}$$

- Without deemphases

$$S_D/N_D = 3D^2 S_x \gamma$$
$$= (3 \times 5^2 \times ½)\gamma = 38\gamma$$

$$\gamma = \frac{S_R}{\eta W}$$

- With deemphases

$$S_D/N_D = (f_\Delta / B_{de})^2 S_x \gamma \approx 640\gamma$$

- Therefore if DSB or SSB system could be exchanged to FM system 640 fold transmission power savings could be achieved. Note, however that the required transmission bandwidth is now about 220 kHz /15 kHz = 15 times larger! Also, a problem is the FM threshold effect that we discuss next.

Comparison of carrier wave modulation systems

Type	$b = B_T/W$	$(S/N)_D/\gamma$	γ_{th}	DC	Complexity	Comments	Typical applications
Baseband	1	1	...	No†	Minor	No modulation	Short-haul links
AM	2	$\frac{\mu^2 S_x}{1+\mu^2 S_x}$	20	No	Minor	Envelope detection $\mu \leq 1$	Broadcast ratio
DSB	2	1	...	Yes	Major	Synchronous detection	Analog data, multiplexing
SSB	1	1	...	No	Moderate	Synchronous detection	Point-to-point voice, multiplexing
VSB	1+	1	...	Yes	Major	Synchronous detection	Digital data
VSB + C	1+	$\frac{\mu^2 S_x}{1+\mu^2 S_x}$	20	Yes‡	Moderate	Envelope detection $\mu < 1$	Television video
PM§	$2M(\phi_\Delta)$	$\phi_\Delta^2 S_x$	$10b$	Yes	Moderate	Phase detection $\phi_\Delta \leq \pi$	Digital data
FM§¶	$2M(D)$	$3D^2 S_x$	$10b$	Yes	Moderate	Frequency detection	Broadcast radio, microwave relay, satellite systems

† Unless direct-coupled.
‡ With electronic DC restoration.
§ $b \geq 2$.
¶ Deemphasis not included.