Bounding Techniques

- Some bounding techniques are investigated in this section in order to serve as a tool for the analysis that follows. We first state the union upper bound that is widely used in the literature.

- Then, some better bounds which are applicable to AWGN channels are also discussed here; they are Gallager bound and sphere upper bound.

The Union Upper Bound

- Consider a code C which consists of M codewords of length n. Let E be an event of decoding error at the output of the decoder, E_i be an event of decoding error given the codeword C_i was transmitted, and $E_{i\to j}$ be an event of decoding error to the codeword C_j when actually C_i was transmitted.

- Let $A_{j,i}$ be the number of codewords, which are at distance $d_j,\ j = 1 \div K,$ from the transmitted codeword C_i. The set $\{A_{j,i}\}$ is defined as the distance spectrum of the code. Because the code may be nonlinear, the set $\{A_{j,i}\}$ may not the same for all transmitted codewords. Generally, it is difficult to calculate $\{A_{j,i}\}$, especially for codes with a large number of codewords.

- It would be wise to have the distance spectrum averaged over all transmitted codewords. Let us define the set $\{A_j\},\ j = 1 \div K,\ A_j = (1/M)\sum_{i=1}^{M} A_{j,i},$ as the average distance spectrum of the code.

- The union upper bound on decoding error reads

$$P(E_i) = P\left(\bigcap_{i\neq j} E_{i\to j}\right)$$
$$\leq \sum_{i\neq j} P(E_{i\to j}).$$

- The probability $P\left(E_{i \to j}\right)$ is often called pairwise error probability (PEP). In an AWGN channel, the PEP $P\left(E_{i \to j}\right)$ depends only on the Euclidean distance d_j between transmitted and decoded codewords, or precisely, $P\left(E_{i \to j}\right) = Q\left(d_j / 2s\right)$, where s^2 is the variance of the additive noise.

- The average decoding error probability for the AWGN channel is written as follows.

$$
\begin{aligned}
\overline{P}(E) &= (1/M)\sum_{i=1}^{M} P(E_i) \\
&\leq \sum_{j=1}^{K} (1/M)\sum_{i=1}^{M} A_{j,i} Q\left(d_j / 2s\right) \\
&= \sum_{j=1}^{K} A_j Q\left(d_j / 2s\right).
\end{aligned}
$$

Some Improved Bounds

- The standard union bound diverges in low signal-to-noise ratio (SNR) region and, particularly, is useless below the channel cut of rate, which is the desired working region for turbo codes and turbo coded modulation.

- Therefore, it is necessary to have new performance bounds for those systems that can account for the noisy working regions. Recently, improved bounds have been presented in (**Duman** and **Salehi**, 1998) and (**Igal** and **Shamai**, 1999).

- **Duman** and **Salehi** proposed a method of bounding turbo codes with maximum-likelihood decoding by using the Gallager bound, (**Gallager**, 1965) or (**Viterbi** and **Omura**, 1979). His bound is tight for a wide range of SNR in which the union bound is already very loose.

- Recently, a better bound for turbo codes was introduced by **Igal** and **Shamai** (1999). The bound was based on the tangential sphere bounding technique which was proven to be tighter than the union bound and the tangential union bound (**Herzberg** and **Poltyrev**, 1996). This bound is also tighter than the bound presented by **Duman** and **Salehi** for most regions of SNR, especially in the severely noisy region.

- In what follows, we state two improved bounding techniques that are applicable to binary codes.

<u>Gallager Bound</u>

- The upper bound for probability of making an error with MLSE decoding when the codeword X_i is transmitted is (**Gallager**, 1965) or

- **Viterbi** and **Omura**, 1979, Section 2.3, 2.4:

$$P(E) \le \sum_{y} \left[P(y \mid X_i)\right]^{1/(1+r)} \left\{ \sum_{j \ne i} \left[P(y \mid X_j)\right]^{1/(1+r)} \right\}^{r}, \; r > 0$$

where $P(y \mid X)$ denotes the conditional pdf of the channel output y given that X is transmitted and r is the optimization parameter.

- When r equals 1, the bound becomes the union bound *derived using Bhattacharya bound*, i.e., that union bound is just a special case of the Gallager bound.

- In order to apply the Gallager bound, the code is first partitioned into subcodes in which each of them contains codewords of same length. The bound was proven to be tighter than the existing union bounds for the range of low SNR.

Sphere Upper Bound

- The bound was proposed in (**Herzberg** and **Poltyrev**, 1994). The sphere upper bound is based on the following central equation

$$P(E) = P(E \mid |\underline{z}| \le r)P(|\underline{z}| \le r) + P(E \mid |\underline{z}| > r)P(|\underline{z}| > r)$$

where E stands for the event of decoding error, $|\cdot|$ represents Euclidean norm, r is a real positive parameter used to optimize, and $\underline{z} = (z_1, z_2, ..., z_n)$ is an $n-D$ additive noise vector.

Because of the inequality $P(E \mid |\underline{\mathbf{z}}| > r) \le 1$, we can write

$$\begin{aligned} P(E) &\le \min_r \{P(E \mid |\underline{\mathbf{z}}|)P(|\underline{\mathbf{z}}| \le r) + P(|\underline{\mathbf{z}}| > r)\} \\ &= \min_r \{P_e(r)\}. \end{aligned}$$

For an AWGN channel, we can write

$$\begin{aligned} P_e(r) &= \sum_{j=1}^{N(r)} A_j P(E_j, |\underline{\mathbf{z}}| \le r) + P(|\underline{\mathbf{z}}| > r) \\ &= \sum_{j=1}^{N(r)} A_j \int_{d_j^2/4}^{r^2} \int_{d_j/2}^{\sqrt{y}} \frac{(y - z_1^2)^{(n-3)/2} \quad e^{-y/2s_N^2}}{\sqrt{p}\, 2^{n/2} s^n \Gamma[(n-1)/2]} \end{aligned}$$

The parameters used in Eq. above are defined as follows

$$y = \sum_{j=1}^{n} z_j^2.$$

$$f_y(y) = \frac{1}{s^n 2^{n/2} \Gamma(n/2)} y^{n/2-1} e^{-y/2s_N^2} u(y).$$

$$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy : \textbf{ gamma function}.$$

$N(r)+\mathbf{1}$ **is the smallest value of** j

so that the condition $r \le d_j/2$ **is satisfied**.

The optimum value of r, denoted by r_0, is found through the following equation

$$\sum_{j=1}^{N(r_0)} A_j \frac{\Gamma(n/2)}{\sqrt{p}\,\Gamma[(n-1)/2]} \int_0^{q_j} \sin^{n-2}(u)du = 1$$

where $d_j/2r_0 = \cos(q_j)$.

For large value of n, the bound can be simplified with invisible loss of tightness as

$$P(E) \le \min_r \left\{ \begin{array}{l} \sum_{j=1}^{N(r)} A_j \left[Q\left(d_j/2s \right) - Q(r/s) \right] \\ \int_0^{r^2 - d_j^2/2} f_{y_1}(y_1) dy_1 \\ + \int_{r^2}^\infty f_y(y) dy \end{array} \right\}$$

where $y_1 = \sum_{j=2}^{n} z_j^2$ and

$$f_{y_1}(y_1) = \frac{1}{s^{n-1} 2^{(n-1)/2} \Gamma[(n-1)/2]} y_1^{(n-3)/2} e^{-y_1/2s^2} u(y_1).$$

Performance Bound for Turbo Codes

- The union upper bound is a popular and effective method of bounding a code performance provided its weight distribution is known.

- Deriving the weight distribution for turbo codes as well as turbo coded modulation, is not tractable for a particular interleaving scheme.

- Therefore the authors in (**Benedetto** and **Montorsi**, 1996a), (**Divsalar** et al., 1995), and (**Duman** and **Salehi**, 1999a) have advanced the idea of forming an average weight function over all possible interleaving schemes.

- The abstract interleaver called *uniform interleaver* was introduced. It is a probabilistic device that maps a given input word into all distinct permutations of it with equal probability.

- The premise is that there is at least one interleaver, which performs better than the average.

- *Binary turbo codes can be considered as block codes if the decoders resort trellis terminations. Consider an* (n,k) *binary block code* C *with rate R=k/n. Denote* h_m *as the minimum distance of the code.*

- Using the upper union bound, we can have a simple conditional upper bound of bit error probability, assuming coherent detection, ML decoding and with channel state information (CSI) illustrated as follows

$$P_b(E \mid r) \leq \sum_{i=1}^{k} \sum_{w=h_m}^{n} \frac{i}{k} A_{i,w}^{C} Q\left(\sqrt{2R(E_b/N_0)\sum_{j=1}^{w} r_j^2} \right)$$

where E_b/N_0 is the ratio of bit energy to noise power density, $A_{i,w}^{C}$ represents the number of codewords with output weight w associated with input weight i and is called input-output weight coefficient (IOWC), and r_j are the multiplicative fading coefficients.

- Note that this bound is applicable to codes on AWGN channels as well as on flat fading channels.

- The only unknown parameter in the equation is the IOWC for turbo codes. The computation of IOWC is intractable for a specific interleaver.

- The authors in (**Benedetto** and **Montorsi**, 1996a) have used the idea of *uniform interleaver* to calculate IOWC averaged over all interleavers while considering fixed component codes.

- ***Uniform interleaver***: A uniform interleaver of the length n is a probabilistic device which maps a given input word of weight i into all distinct $\binom{n}{i}$ permutations of it with equal probability $1/\binom{n}{i}$.

- Assume $A_{i,w}^{C_1}$ and $A_{i,w}^{C_2}$ are the IOWC's of component codes, CC-1 and CC-2 respectively. Using the concept of uniform interleaver, IOWC for turbo codes can be calculated as

$$A_{i,w}^{C} = \sum_{w_1, w_2 : w_1 + w_2 = w} \frac{A_{i,w_1}^{C_1} A_{i,w_2}^{C_2}}{\binom{N}{i}}.$$

Define B_w^C as

$$B_w^C = \sum_{i=1}^{k} \frac{i}{k} A_{i,w}^{C}.$$

- Now, we are ready to calculate the conditional probability $P_b(E \mid r)$. We can obtain $P_b(E)$ by averaging over all r_j assuming that the fading is slow

$$\overline{P}_b(E) = \int_r p(r) P_b(E \mid r) dr$$

$$\leq \sum_{w=h_m}^{n} B_w^C \int_r p(r) Q\left(\sqrt{2R(E_b/N_0) \sum_{j=1}^{w} r_j^2} \right) dr.$$

By simplifying the above equation some versions of the bit error probability bounds, for example in (**Hall** and **Wilson**, 1998) were obtained.

Performance Bound for Turbo Coded Modulation

- Performance analysis of turbo coded modulation schemes requires more evaluations than their counterparts, turbo codes, due to the nonlinearity of the modulation process, in general.

- In the case of turbo coded modulation, we have to investigate all the possible transmitted codewords than assuming that the all-zero codeword was transmitted as being done for turbo codes.

- Due to the inherent presence of interleaver(s) of the turbo coded modulation schemes, the investigation of all possible transmitted codewords is very undesirable, if not possible, especially when the size of the interleaver is large.

- The performance of turbo coded modulation schemes can be achieved when the distance spectra are available.

- Since it is not plausible to find the exact distance spectrum for a particular interleaver of the underlying turbo code, it is often done with the idea of *uniform interleaver* averaged over all the possibilities of interleaver choices.

- The first effort to get the performance bound was made by **Duman** and **Salehi** (1999a). In their analysis, the two abstract interleavers were also added after the underlying turbo code (before mapping) to keep the derivation tractable. The distance profile was introduced.

- Unfortunately, the actual bound can not be found, but instead the true upper bound and average upper bound were given.

- In this section, we will review the previous work, mostly from (**Duman** and **Salehi**, 1999a). We focus only on the case of 2 bits/sec/Hz 16-QAM and the underlying turbo code with known weight distribution.

- We denote a codeword of a turbo coded modulation scheme is a group of N_S channel symbols. Each codeword corresponds to the unique label sequence $X_i = \left(X_i^1, X_i^2, ..., X_i^{N_S}\right)$ before the mapping (modulation function).

- Let the size of the interleaver is N. We have the relation $N = 2N_S$ for the 2 bits/sec/Hz 16-QAM schemes. Each channel symbol corresponds to 4-bit group, denoted by $X_i^j = \left(s_{i,1}^j s_{i,2}^j ; p_{i,1}^j p_{i,2}^j\right); j = 1 \div N_S, i = 1 \div 2^N$; where $\left(s_{i,1}^j s_{i,2}^j\right)$ is a pair systematic bits and $\left(p_{i,1}^j p_{i,2}^j\right)$ is a pair of parity bits.

- When certain label sequence, say $X_i = \left(X_i^1, X_i^2, ..., X_i^{N_S}\right)$, was transmitted, the decoded label sequence at the output of the decoder is $Y_i = \left(Y_i^1, Y_i^2, ..., Y_i^{N_S}\right)$.

- An error sequence E_i is defined as the difference between the transmitted and decoded label sequences. Equivalently we can write $E_i = X_i \oplus Y_i$; where $\oplus$ stands for modulo-2 sequence addition.

- Due to the linearity of the underlying turbo code, any error sequence, E_i, is also a label sequence that represents a certain codeword, say $j(E_i)$, where $j(\cdot)$ is the one-to-one mapping function.

- Define the error sequence of type **n** as a vector containing all parameters $n_{i,w}$'s which stand for the total number of channel symbols which have i systematic bits and w parity bits in error.

- The possible combinations of (i,w) can be $\{(0,1);(0,2);(1,0);(1,1);(1,2);(2,0);(2,1);\ (2,2)\}$.

- An error sequence can be completely represented by **n**. The number of error sequences represented by **n** is $f(\mathbf{n})$. The expected value of this function, denoted by $\overline{f}(\mathbf{n})$, was found to be (**Duman** and **Salehi**, 1999a)

$$\overline{f}(\mathbf{n}) = \sum_{i,w} \overline{A}_{i,w} \frac{P_{i,w,\mathbf{n}}}{\binom{N}{i}\binom{N}{w}}$$

where $\overline{A}_{i,w}$ is the average number of codewords of the underlying turbo code with information weight i and parity weight w, and $P_{i,w,\mathbf{n}}$ is defined as follows

$$P_{i,w,\mathbf{n}} = \begin{cases} \binom{N_S}{n_{0,1},\ldots,n_{2,2}} 2^{n_{0,1}} 2^{n_{1,0}} 2^{2n_{1,1}} 2^{n_{1,2}} 2^{n_{2,1}}, \\ \textbf{if} \quad \begin{cases} n_{1,0}+n_{1,1}+n_{1,2}+2n_{2,0}+2n_{2,1}+2n_{2,2}=i \\ \textbf{and} \\ n_{0,1}+2n_{0,2}+n_{1,1}+2n_{1,2}+n_{2,1}+2n_{2,2}=w \end{cases} \\ 0, \qquad \textbf{elsewhere.} \end{cases}$$

- The error sequence E_i is random in nature. For a specific E_i, the Euclidean distance between the pair of transmitted and decoded codewords that results in E_i is also random, or more exactly, it is a discrete random variable. This is due to the nonlinearity of the mapping process.
- We have the definition of *distance profile*, denoted by $D_{\mathbf{n}}$, as a $\mathbf{2}\times k_{\mathbf{n}}$ matrix to specify this discrete random variable

$$D_{\mathbf{n}} = \begin{bmatrix} d^2_{\mathbf{n},1} & d^2_{\mathbf{n},2} & \cdots & d^2_{\mathbf{n},k_{\mathbf{n}}} \\ p_1 & p_1 & \cdots & p_{k_{\mathbf{n}}} \end{bmatrix}.$$

- The expected number of error sequences represented by the vector **n** is $2^N \overline{f}(\mathbf{n})$. The expected number of error sequences represented by **n** that result in an Euclidean distance of d_i is $2^N \overline{f}(\mathbf{n}) p_i$.

- The union bound for bit error probability averaged over all interleavers can be written as

$$\bar{P}_b(E)) \le E\left[\sum_{E_i} \frac{i_{E_i}}{N} 2^{-N} Q\left(\sqrt{d_{E_i}^2 / 2N_0}\right)\right]$$

$$= \sum_{i=1}^{N} \sum_{w=0}^{N} \sum_{n_{0,1}} \cdots \sum_{n_{2,2}} \sum_{k=1}^{k_n} \frac{i}{N} \bar{f}(\mathbf{n}) p_k Q\left(\sqrt{d_{\mathbf{n},k}^2 / 2N_0}\right)$$

where $E[\cdot]$ is the expectation taken with respect to all the possible interleavers, i_{E_i} is the number of information bits in error contained in the codeword corrupted by the error sequence E_i, d_{E_i} is the Euclidean distance caused by the error sequence E_i, N_0 is the one-sided noise power spectrum density.

Spherical Performance Bounds for Turbo Coded Modulation Schemes over AWGN Channels

- For the multilevel coded modulation, **Burr** (1997) has presented the ensemble bound that is only applicable to schemes using M-PSK modulation. The bound does not account for any specific codes with given distance spectra, but instead, it deals with the family of codes with same decoding delay (block length).

- It has been shown that the sphere upper bound (**Herzberg** and **Poltyrev**, 1994) is always tighter than the conventional union bound, especially for a region of low signal to noise ration.

- This point naturally hints us to apply the Spherical bounding method to investigate the performance of (turbo) coded modulation schemes using any kinds of modulation (including M-QAM).

- We are also interested in finding bounds that can take into account the distance spectra of the underlying binary turbo codes.

- This part investigates the performance of turbo coded modulation schemes over AWGN channels. The presented bounds are based on the spherical bounding technique. In the derivation, Euclidean geometry is used as a tool to get the bound. The bound accounts for all kinds of modulations as well as specific distance spectra of the underlying turbo codes.

- We start our analysis from the equation (as discussed earlier) $P_e(r) = \sum_{j=1}^{N(r)} A_j P\left(E_j, |\underline{\mathbf{z}}| \le r\right) + P\left(|\underline{\mathbf{z}}| > r\right)$. We now need to calculate or upper bound the joint probability $P\left(E_j, |\underline{z}| \le r\right)$.

- For the codes of length n using two-dimension modulation schemes, each codeword can be represented as a unique vector in the $2n-D$ Euclidean space. The additive noise is then also a vector in this space. The geometric location of the intersection $E_j \cap \{\underline{z} : |\underline{z}| \le r\}$, denoted by D_j^{2n}, for a certain transmitted codeword C_i is the crosshatched region shown in the Fig., for the case of two-dimensional space.

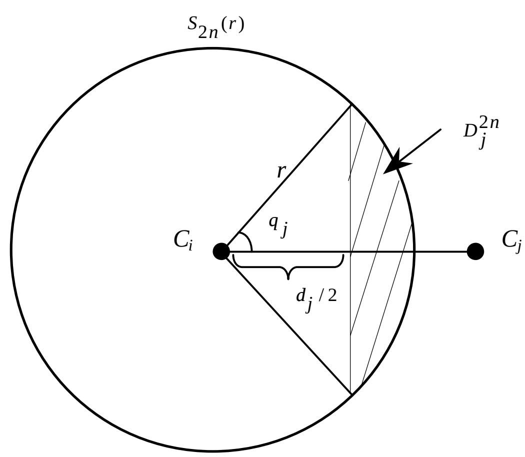

Figure Illustration of the region of integration

Clearly, we calculate the probability $P\left(E_j, |\underline{z}| \le r\right)$ as

$$P\left(E_j, |\underline{z}| \le r\right) = \int_{\underline{z} \in D_j^{2n}} p_Z(\underline{z})\, dV$$

where dV is the element of volume and $p_z(\underline{z})$ is the circular probability density of the noise vector.

- Now we can simplify this by extending the region of integration above. Define $D_{j,ext}^{2n}$ as the region of the ring-shaped made by the intersections between the $2n-D$ cone of half angle q_j and two *2n-D* spheres of radii $d_j/2$ and *r*. This is illustrated in Fig. below for the case of two dimensional space.

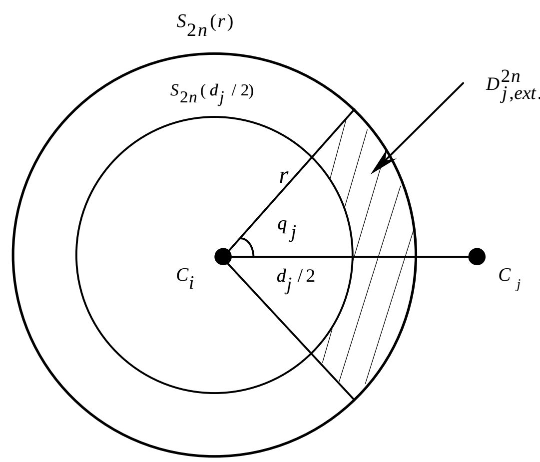

Figure The extended region of integration

Clearly, $D_j^{2n} \subset D_{j,ext.}^{2n}$, so we can upper bound $P(E_j, |\underline{z}| \leq r)$ as

$$P(E_j, |\underline{z}| \leq r) \leq \int_{\underline{z} \in D_{j,ext.}^{2n}} p_z(\underline{z}) dV .$$

- In the AWGN channel, the noise perturbs all coordinates normally and independently with variance s^2. It produces a spherical Gaussian distribution in the $2n-D$ space (**Sloane** and **Wyner**, 1993). The probability of its moving the signal point a distance *x* is given by

$$\frac{1}{\left(2ps^{2}\right)^{n}} e^{-x^{2}/2s^{2}} dV.$$

The differential volume of the ring-shaped region is $xdxdq$, where q span from 0 to q_i radian, times the surface of the $2n-D$ sphere of radius $x\sin(q)$ in $(2n-1)-D$ space illustrated in the Fig. above; that is

$$xdxdq \frac{(2n-1)p^{n-1/2}(x\sin q)^{2n-2}}{\Gamma(n+1/2)}.$$

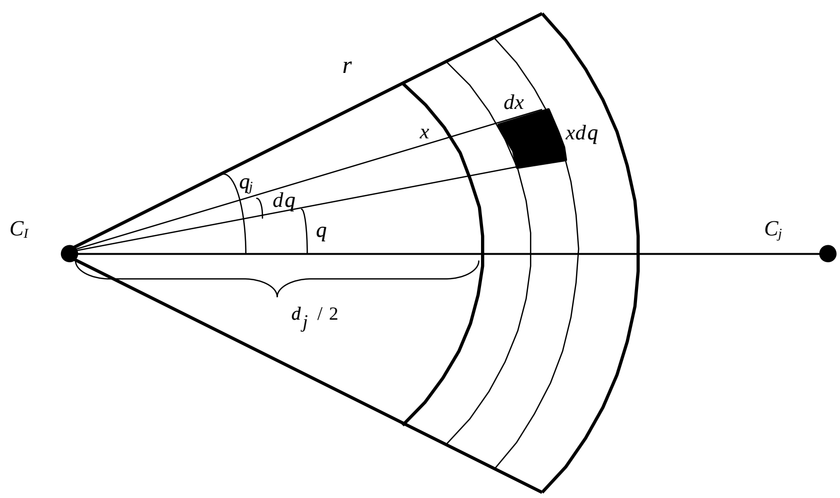

Figure The differential volume that causes differentially conditional error

So, the differential probability for the ring-shaped region is

$$\frac{1}{\left(2ps^{2}\right)^{n}} e^{-x^{2}/2s^{2}} xdxdq \frac{(2n-1)p^{n-1/2}(x\sin q)^{2n-2}}{\Gamma(n+1/2)}$$

$$= \frac{2n-1}{\left(\sqrt{2}s\right)^{2n-1}\Gamma(n+1/2)} \frac{x^{2n-1}e^{-x^{2}/2s^{2}}dx}{\sqrt{2p}s}(\sin q)^{2n-2}dq.$$

Now, we are ready to upperbound $P\left(E_j, |z| \le r\right)$ as

$$P\left(E_j, |z| \le r\right) \le \int\limits_{\underline{z} \in D_{j,ext.}^{2n}} p_z(\underline{z}) dV$$

$$= \frac{2n-1}{\left(\sqrt{2}s\right)^{2n-1} \Gamma(n+1/2)} \int\limits_{d_j/2}^{r} x^{2n-1} \frac{e^{-x^2/2s^2}}{\sqrt{2p}s} dx$$

$$\int\limits_{0}^{q_j} (\sin q)^{2n-2} dq.$$

The probability $P\left(|z| > r\right)$ is calculated as

$$P\left(|z| > r\right) = \int\limits_{r^2}^{\infty} f_y(y) dy$$

where $f_y(y) = \dfrac{1}{s^{2n} 2^n \Gamma(n)} y^{n-1} e^{-y/2s^2} u(y).$

Our final result for the bound of decoding error is

$$P(E) \le \min_r \left\{ \begin{array}{l} \sum\limits_{j=1}^{N(r)} A_j \dfrac{2n-1}{\left(\sqrt{2}s\right)^{2n-1} \Gamma(n+1/2)} \cdot \int\limits_{d_j/2}^{r} x^{2n-1} \dfrac{e^{-x^2/2s^2}}{\sqrt{2p}s} dx \\ \qquad \int\limits_{0}^{q_j} (\sin q)^{2n-2} dq \\ + \int\limits_{r^2}^{\infty} f_y(y) dy \end{array} \right\}.$$

- The optimal value of r, denoted by r_0, is found by differentiating the right hand side of Eq. and equate it to zero, i.e. r_0 is the root of the following equation

$$\sum_{j=1}^{N(r)} A_j \frac{2n-1}{\left(\sqrt{2}s\right)^{2n-1}\Gamma(n+1/2)} r^{2n-1} \frac{e^{-x^2/2s^2}}{\sqrt{2p}s} \int_0^{q_j} (\sin q)^{2n-2} dq = 2rf_y\left(y = r^2\right).$$

We now can write the performance bound for bit error probability of the turbo coded modulation scheme as

$$\overline{P}_b(E) \le \min_r \left\{ \sum_{j=1}^{N(r)} B_j \frac{2n-1}{\left(\sqrt{2}s\right)^{2n-1}\Gamma(n+1/2)} \int_{d_j/2}^{r} x^{2n-1} \frac{e^{-x^2/2s^2}}{\sqrt{2p}s} dx \int_0^{q_j} (\sin q)^{2n-2} dq + \int_{r^2}^{\infty} f_y(y) dy \right\}$$

where $n = N_S$ for TTCM and $B_j = \sum_{i=1}^{N} \sum_{w=0}^{N} \sum_{n_{0,1}} \cdots \sum_{n_{2,2}} \frac{i}{N} \overline{f}(\mathbf{n}) p_j$ with $N = 2N_S$ for the 2 bits/sec/Hz 16-QAM scheme.

- When the coded modulation scheme uses an M-PSK modulation scheme, we can have the more-easily-evaluated ensemble bound by employing the fact that all the codewords have the same energy of $N_S E_S$, where E_S is the energy of each M-PSK constellation point.

- In that case all the codewords lie on the same sphere of radius $\sqrt{N_S E_S}$ in the $2N_S - D$ space. Basing on the random coding argument (**Sloane** and **Wyner**, 1993), we can have at least one code that performs as well as the random code. The average performance bound of a spherical code in the $2N_S - D$ space can be found as (**Burr**, 1997) and (**Aldis**, 1992).

$$P(E) \le \int_0^{p} p(q)\left[1 - \left(1 - P_{2N_S}(q)\right)^{V-1}\right] dq$$

where V is the number of possible codewords. The other variables used in Eq. are defined below

$$p(q) = \frac{(2N_S - 1)\Gamma(N_S)}{\sqrt{p}\,\Gamma(N_S + 1/2)} \sin^{2N_S - 2}(q) \int_0^{\infty} r^{2N_S - 1} \exp\left(-\frac{r^2 - 2r\sqrt{N_S E_S}\cos(q) + N_S E_S}{N_0} \right) dr$$

where N_0 is the additive noise power spectrum density.

$$P_{2N_S}(q) = \frac{(2N_S - 1)\Gamma(N_S + 1)}{\sqrt{p}\,\Gamma(N_S + 1/2)} \int_0^{q} \sin^{2N_S}(a)\,da\,.$$

Discussion

- A new upper bound has been proposed for the turbo coded modulation schemes over AWGN channels.

- Our goal is to derive a tight bound for turbo coded modulation schemes over AWGN channels, especially when they work in the region of low signal to noise ratio.

- Although we have reduced the tightness of the bound by using the extended region of integration instead of the exact one we expect that this loss of tightness is negligible. This implies that the derived bound is still better than the conventional union bound.

- This is a reasonable expectation, especially when the block length (or the interleaver size) of the code becomes larger. In order to evaluate the bound, we need to use numerical tables.

- When the block length is large, the numerical evaluation of the bounds becomes burdensome. The bound can be applied to other kinds of coded modulation schemes; such as trellis coded modulation with trellis termination, block coded modulation schemes.

Performance of Turbo Coded Modulation over Frequency Selective Rayleigh Fading Channels

- It has been considered difficult to upper bound turbo coded modulation even for AWGN channels because, in general, we cannot use the uniform error properties here (**Benedetto** and **Montorsi**, 1996a).

- Another reason is due to the large interleaver size of the underlying turbo code that makes the derivation of the code distance spectrum challenging. On flat fading channels, **Duman** and **Salehi** (1999b) have presented the union bounds for turbo coded modulation schemes. In keeping the derivation tractable, uniform interleaver concept was also used.

- To the best of our knowledge, performance bounds for turbo coded modulation over frequency selective Rayleigh fading channels are still non-existent in the literature.

- Therefore, there is a need for analytical performance bounds on bit error probability of this scheme over these channels, especially when the decoder operates only on the code states, i.e., the receiver is mismatched to the channels. In (**Carlisle** et al., 1994), mismatched decoding has been analyzed for trellis coded modulation, but the considered channel was the static ISI one.

- One question raises our concern "does a receiver for turbo coded modulation designed for flat fading channel work well over frequency selective Rayleigh fading channels?"

- This section investigates the performance of turbo coded modulation over the frequency selective Rayleigh fading channels for the worst case when the receiver only operates on the code states.

- We are particularly interested in a bound that accounts for raw ISI. We derive an approximate performance bound for the turbo coded modulation scheme using square M-QAM when the receiver uses an MLSE decoder for frequency selective Rayleigh fading channels.

- Our evaluation is useful because we do not need to consider the channel states, especially when the number of channel states is large. Since we are interested in finding the average bound, the following simple assumptions are made: 1) the channel symbols are independent of each other, 2) they are equally likely.

- The first assumption becomes more acceptable when we are using an interleaver of reasonable depth after the encoder. The other one is particularly true when the number of uncoded bits exceeds that of coded bits. We later see that those assumptions are not so optimistic, especially when the bound is applied to TCM.

- The system model we will investigate is shown in Fig. below. The information sequence $\{u_k\}$ is encoded by a coded modulation encoder, which can be TCM, or TTCM. The output symbol sequence $\{c_k\}$, where c_k is taken from one of M possible values, is block interleaved to break up burst errors caused by the fading channel before being transmitted into the channel.

- We assumed that the interleaving depth is infinite, so that the fading is uncorrelated. In reality, even this assumption cannot be met, by proper use of interleaving depth, which is greater than the maximum fade duration, the correlation is negligible.

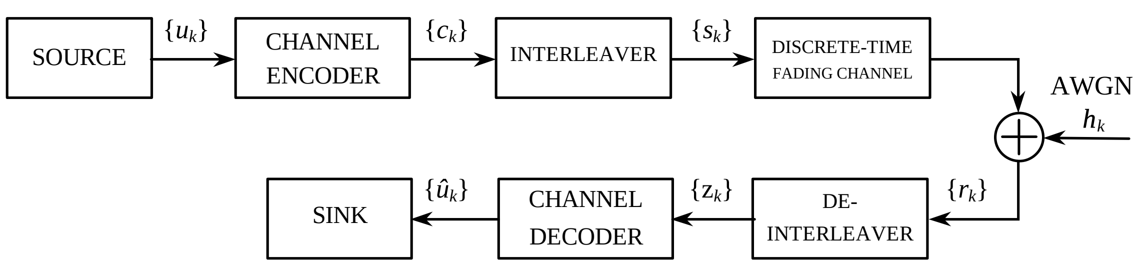

Figure System model

- Previously we know that the transmit filter, multipath fading channel, and receive filter can be modeled as an equivalent discrete-time channel which consists of $L+1$ taps (paths). The output of the equivalent channel, denoted by r_k, can be written as

$$
\begin{aligned}
r_k &= \sum_{i=0}^{L} g_i(k) s_{k-i} + h_k \\
&= g_0(k) s_k + \sum_{i=1}^{L} g_i(k) s_{k-i} + h_k \\
&= g_0(k) s_k + z_k + h_k
\end{aligned}
$$

where $z_k = \sum_{i=1}^{L} g_i(k) s_{k-i}$ is the ISI component, h_k is the additive Gaussian noise with variance s_h^2 on each dimension. From Eq., we can have the equivalent model for the ISI channel as in Fig. here.

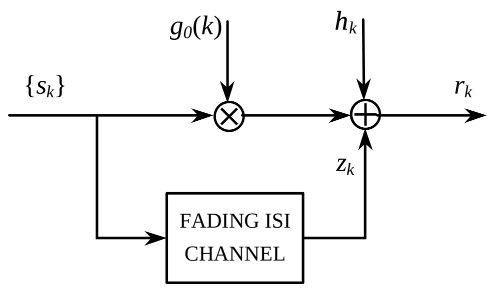

Figure An equivalent fading ISI channel

We now find the pdf of ISI component, z_k,

$$
\begin{aligned}
z_k &= \sum_{i=1}^{L} g_i(k) s_{k-i} = \sum_{i=1}^{L} z_{k,i} \\
&= \sum_{i=1}^{L} \left[g_i^{\text{Re.}}(k) + j g_i^{\text{Im.}}(k) \right] \left(s_{k-i}^{\text{Re.}} + j s_{k-i}^{\text{Im.}} \right) \\
&= \sum_{i=1}^{L} \left\{ \begin{array}{l} \left[g_i^{\text{Re.}}(k) s_{k-i}^{\text{Re.}} - g_i^{\text{Im.}}(k) s_{k-i}^{\text{Im.}} \right] + \\ j \left[g_i^{\text{Re.}}(k) s_{k-i}^{\text{Im.}} + g_i^{\text{Im.}}(k) s_{k-i}^{\text{Re.}} \right] \end{array} \right\}.
\end{aligned}
$$

- The variables in Eq. (3.42) with the superscripts Re. and Im. imply the real and imaginary parts of them respectively. For brevity, we need to investigate the complex-valued random variable $z_{k,i}=(ac-bd)+j(ad+bc)$, where a, b, c, and d represent $g_i^{\mathrm{Re.}}(k)$, $g_i^{\mathrm{Im.}}(k)$, $s_{k-i}^{\mathrm{Re.}}$, and $s_{k-i}^{\mathrm{Im.}}$ respectively. The pdf's of a, b, c, and d can be found easily as follows (**Papoulis**, 1984)

$$p(a)=\frac{1}{\sqrt{2\pi}\,\sigma_{a,i}}\exp\left(-\frac{a^2}{2\sigma_{a,i}^2}\right).$$

$$p(b)=\frac{1}{\sqrt{2\pi}\,\sigma_{a,i}}\exp\left(-\frac{b^2}{2\sigma_{b,i}^2}\right).$$

$$p(c)=\sum_{i=1}^{\sqrt{M}} p_i\,\delta(c-c_i).$$

$$p(d)=\sum_{i=1}^{\sqrt{M}} q_i\,\delta(d-d_i).$$

where $\sigma_{a,i}^2$ **and** $\sigma_{b,i}^2$ are variances of the inphase and quadrature components of the i^{th} fading path; M ($\sqrt{M}$ integer) is the number of the constellation points (M-QAM); p_i **and** q_i are the probabilities of the variables c and d taking on values c_i **and** d_i respectively.$\{c_i\}$ and $\{d_i\}$ are the sets of possible values of the inphase and quadrature components of the constellation.

- Using the characteristic functions, the pdf of the variable $z_{k,i}$, denoted by $p(z_{k,i})$, can be found as follows

$$p(z_{k,i}) = \sum_{k=1}^{\sqrt{M}}\sum_{j=1}^{\sqrt{M}} p_k q_j \frac{1}{\sqrt{2p}\, s_i \left(c_k^2 + d_j^2\right)^{1/2}} \exp\left(-\frac{\left(z_{k,i}^{\mathrm{Re.}}\right)^2}{2s_i^2\left(c_k^2 + d_j^2\right)} \right)$$

$$+ j\sum_{k=1}^{\sqrt{M}}\sum_{j=1}^{\sqrt{M}} p_k q_j \frac{1}{\sqrt{2p}\, s_i \left(c_k^2 + d_j^2\right)^{1/2}} \exp\left(-\frac{\left(z_{k,i}^{\mathrm{Im.}}\right)^2}{2s_i^2\left(c_k^2 + d_j^2\right)} \right)$$

$$= \sum_{k=1}^{M} Q_k \frac{1}{\sqrt{2p}\, s_i |s_k|} \exp\left(-\frac{\left(z_{k,i}^{\mathrm{Re.}}\right)^2}{2s_i^2 |s_k|^2} \right)$$

$$+ j\sum_{k=1}^{M} Q_k \frac{1}{\sqrt{2p}\, s_i |s_k|} \exp\left(-\frac{\left(z_{k,i}^{\mathrm{Im.}}\right)^2}{2s_i^2 |s_k|^2} \right)$$

where Q_k is the probability of the magnitude of the channel symbol (constellation point) taking on the value $|s_k|$ and s_i^2 is the variance of fading path *i* assuming that its inphase and quadrature components have the same distribution.

From Eqs. we can have the pdf of the ISI component z_k, denoted by $p(z_k)$, as

$$p(z_k) = \sum_{k_1=1}^{M} \cdots \sum_{k_L=1}^{M} \left\{ \prod_{j=1}^{L} Q_{k_j} \times \frac{1}{\sqrt{2p}\left(\sum_{i=1}^{L} s_i^2 |s_{k_i}|^2 \right)^{1/2}} \exp\left(-\frac{\left(z_k^{\mathrm{Re.}}\right)^2}{2\sum_{i=1}^{L} s_i^2 |s_{k_i}|^2} \right) \right\}$$

$$+ j\sum_{k_1=1}^{M} \cdots \sum_{k_L=1}^{M} \left\{ \prod_{j=1}^{L} Q_{k_j} \times \frac{1}{\sqrt{2p}\left(\sum_{i=1}^{L} s_i^2 |s_{k_i}|^2 \right)^{1/2}} \exp\left(-\frac{\left(z_k^{\mathrm{Im.}}\right)^2}{2\sum_{i=1}^{L} s_i^2 |s_{k_i}|^2} \right) \right\}.$$

The variance of z_k, denoted by $s_{z_k}^2$, can be found as

$$s_{z_k}^2 = E_{z_k}\left[|z_k - \bar{z}_k|^2 \right]$$

$$= 2\sum_{k_1=1}^{M} \cdots \sum_{k_L=1}^{M} Q_{k_1} \times \cdots \times Q_{k_L} \left(s_1^2 |s_{k_1}|^2 + \cdots + s_L^2 |s_{k_L}|^2 \right)$$

$$= 2\sum_{k_1=1}^{M} \cdots \sum_{k_L=1}^{M} \left[\prod_{j=1}^{L} Q_{k_j} \sum_{i=1}^{L} s_i^2 |s_{k_i}|^2 \right].$$

- Using the assumption that channel symbols are equally likely, $s_{z_k}^2$ can be written as

$$s_{z_k}^2 = \frac{2}{M^L} \sum_{k_1=1}^{M} \cdots \sum_{k_L=1}^{M} \sum_{i=1}^{L} s_i^2 \left| s_{k_i} \right|^2 .$$

If all paths have the same distribution with variance s^2, $s_{z_k}^2$ then reduces to

$$s_{z_k}^2 = \frac{2L s^2}{M} \sum_{k=1}^{M} \left| s_{k_i} \right|^2$$
$$= 2L s^2 \overline{\mathbf{E}}_{\mathbf{s}}$$

where $\overline{E}_s$ is the average power of the channel symbol.

- From Eq., we observe that the distribution of the ISI term is generally not Gaussian due to the discrete distribution of the channel symbols, s_k**'s**.

- We still assume that the distribution of z_k can be approximately shaped to the complex-valued Gaussian distribution with the same mean and variance; that is z_k has the $N(0, s_{z_k} / \sqrt{2})$ distribution for its real and imaginary components.

- Note that for the case of 4-QAM and with the assumption that all L paths (excluding the first one) have the same distribution, the distribution of z_k exactly follows the Gaussian one.

- With the above assumptions, we can have the equivalent channel output

$$r_k = g_0(k)s_k + z_k + h_k$$
$$= g_k s_k + n_k$$

where $g_k = g_0(k)$ **and** $n_k = z_k + h_k$ whose real and imaginary parts follow $N\left(0, \sqrt{s_h^2 + s_{z_k}^2 / 2}\right)$.

- Finally, Eq. above exactly brings our analysis to the case of flat-fading channel which has been studied in (**Duman** and **Salehi**, 1999b) and (**Tellambura**, 1996), for example.

<u>Results</u>

- Figure 1 gives us an example of ISI distributions of 16-QAM scheme over frequency selective Rayleigh fading channels with 3 and 4 paths (this is equivalent to the ISI components having 2 and 3 paths respectively).

- For the 2 path ISI case, variances of the two fading paths are the same, and equal 20% of the total fading gain. For the 3 path ISI case, all 4 paths (including the first desired path) have the same variance. We can see that the ISI pdf's are very close to the equivalent Gaussian distributions. Note that the average channel symbol power equals 10 in this drawing.

- Figure 2 shows the average union upper bounds for turbo coded modulation using 16-QAM with two different mapping schemes, A and B as in (**Duman** and **Salehi**, 1999b).

- The bounds are plotted for the interleaver of length 200. The multipath channel has three paths. The ISI component consists of 7% of the total fading gain. We also show the simulation results with sub-optimal iterative decoding.

- As expected, the performance of the scheme for mapping B is better than that for mapping A. For the low SNR range, differences between the bounds and simulation results are quite significant. This is due to the contribution of the union bounding technique, which is relatively loose for the low SNR range.

- We also observe that in all cases the simulation results (at the 6th iteration) are worse than the upper bounds. We believe that this is due to the sub-optimal iterative decoding.

- Figure 3 illustrates the true union bound of 4-QAM TCM (**Tellambura**, 1996). The fading channel also has 3 paths and the ISI component consists of only 5% of the total fading gain since TCM does not perform well with high level of raw ISI. We can see that the approximate bound is significantly tight for large SNR.

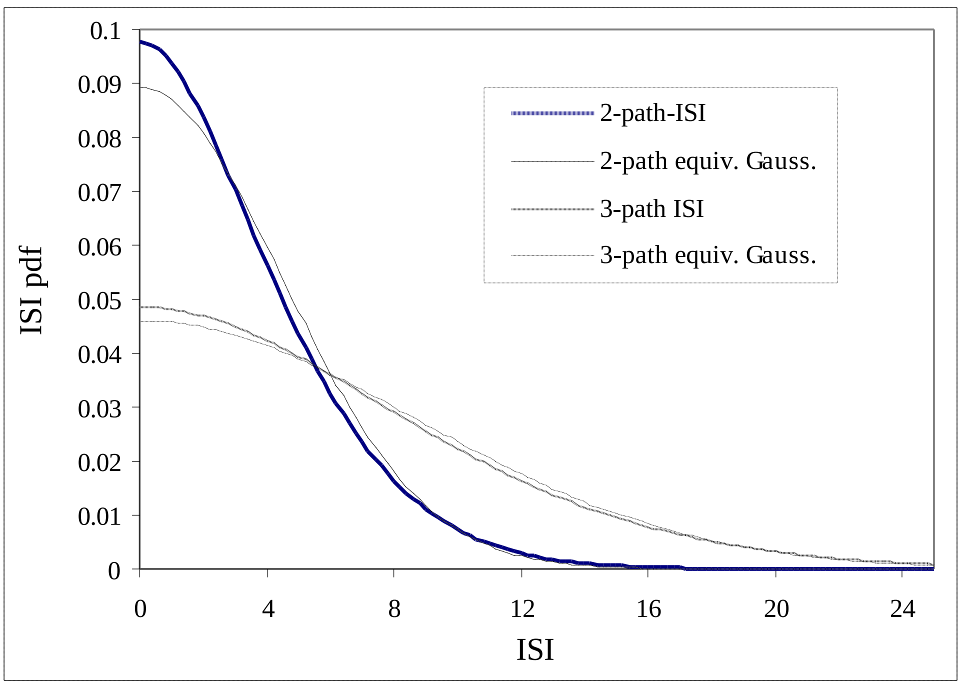

Figure 1 ISI distributions with different numbers of paths

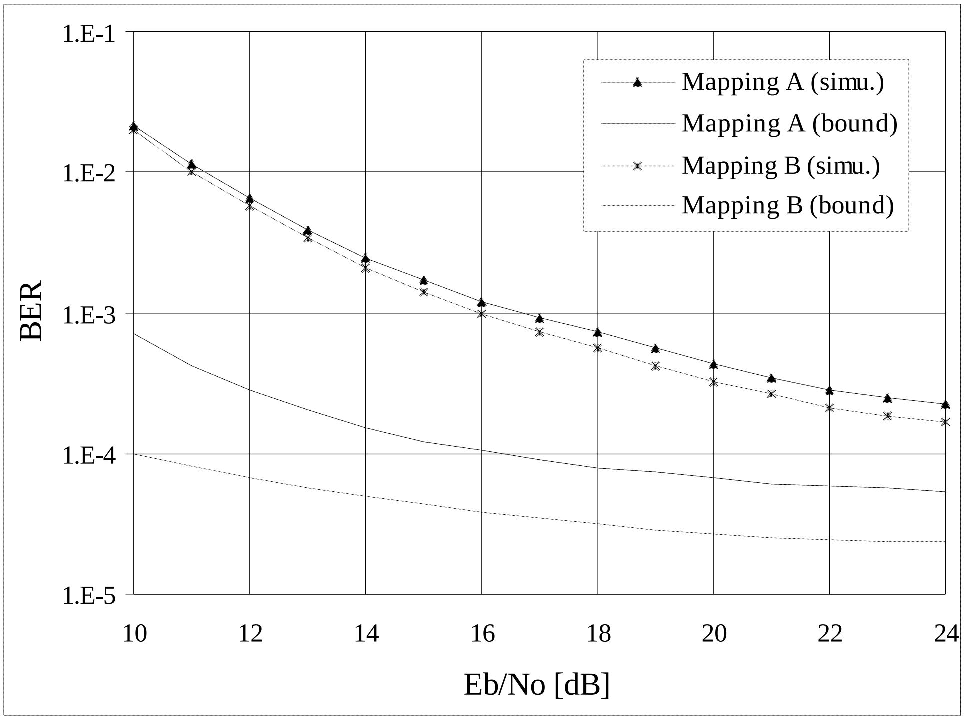

Figure 2 Bounds for 16-QAM turbo coded modulation with 5/7 component codes and interleaver length N=200

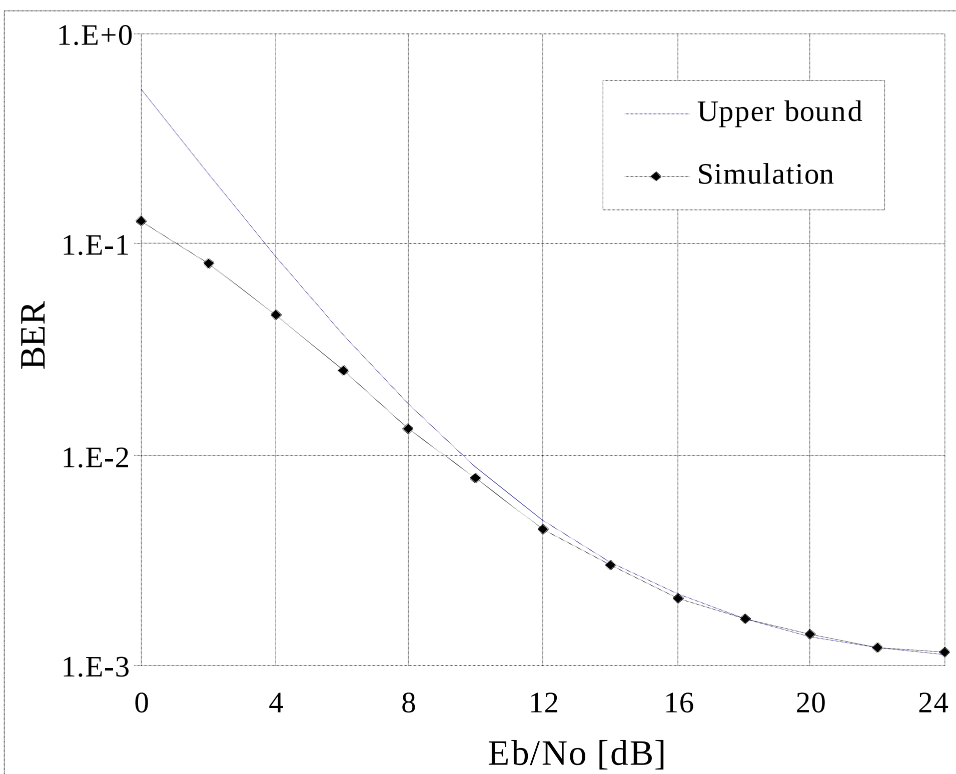

Figure 3 Bound for 4-QAM trellis coded modulation with 2-state component code and block length N=200

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