S-72.311 ADVANCED ERROR CONTROL SCHEMES

Home assignment #1, submission date: March 13th 2001

1. (Van Trees, problem 2.2.1) Consider the following binary hypothesis testing problem:

$$H_1: \quad r = s + n$$
$$H_0: \quad r = n,$$

where s and n are independent random variables.

$$p_s(S) = ae^{-aS} \qquad S \geq 0,$$
$$0 \qquad S < 0,$$
$$p_n(N) = be^{-bN} \qquad N \geq 0,$$
$$0 \qquad N < 0.$$

a) Prove that the likelihood ratio test reduces to

$$\text{choose } H_1 \text{ if } \quad R > \gamma$$
$$\text{choose } H_0 \text{ if } \quad R < \gamma.$$

b) Find γ for the optimum Bayes test as a function of the costs and a priori probabilities.

c) Now assume that we need a Neyman-Pearson test. Find γ as a function of P_F, where

$$P_F \triangleq \Pr[\text{say } H_1 \mid H_0 \text{ is true}].$$

2. (Van Trees, problem 2.2.6) The observation r is defined in the following manner:

$$r = bm_1 + n \quad : H_1,$$
$$r = n \quad : H_0,$$

where b and n are independent zero-mean Gaussian variables with variances σ_b^2 and σ_n^2 respectively.

a) Find the LRT and draw a block diagram of the optimum processor.

b) Draw the ROC.

c) Assume that the two hypotheses are equally likely. Use the criterion of minimum probability of error. What is the $\Pr(\varepsilon)$?

3. (Van Trees, problem 2.2.12, parts a and b) Randomized tests. Our basic model of the decision problem in the text (p. 24) did not permit randomized decision rules. We can incorporate them by assuming that at each point $\mathbf{R}$ in Z we say H_1 with probability $\phi(\mathbf{R})$ and say H_0 with probability $1-\phi(\mathbf{R})$. The model in the text is equivalent to setting $\phi(\mathbf{R})=1$ for all $\mathbf{R}$ in Z_1 and $\phi(\mathbf{R})=0$ for all $\mathbf{R}$ in Z_0.

a) We consider the Bayes criterion first. Write the risk for the above decision model.

b) Prove that a LRT minimizes the risk and a randomized test is *never* necessary.

4. (Van Trees, problem 2.2.13) The random variable $\Lambda(\mathbf{R})$ is defined by (13) and has a different probability density on H_1 and H_0. Prove the following:

a) $E[\Lambda^n \mid H_1] = E[\Lambda^{n+1} \mid H_0]$,

b) $E[\Lambda \mid H_0] = 1$,

c) $E[\Lambda \mid H_1] - E[\Lambda \mid H_0] = \mathrm{var}[\Lambda \mid H_0]$.

5. (Van Trees, problem 2.2.14) Consider the random variable Λ. In (94) we showed that

$$p_{\Lambda|H_1}(X \mid H_1) = X p_{\Lambda|H_0}(X \mid H_0).$$

a) Verify this relation by direct calculation of $p_{\Lambda|H_1}[\cdot]$ and $p_{\Lambda|H_0}[\cdot]$ for the densities in Example 1 [p. 27, (19) and (20)].

b) On page 37 we saw that the performance of the test in Example 1 was completely characterized by d^2. Show that

$$d^2 = \ln[1 + \mathrm{var}(\Lambda \mid H_0)].$$