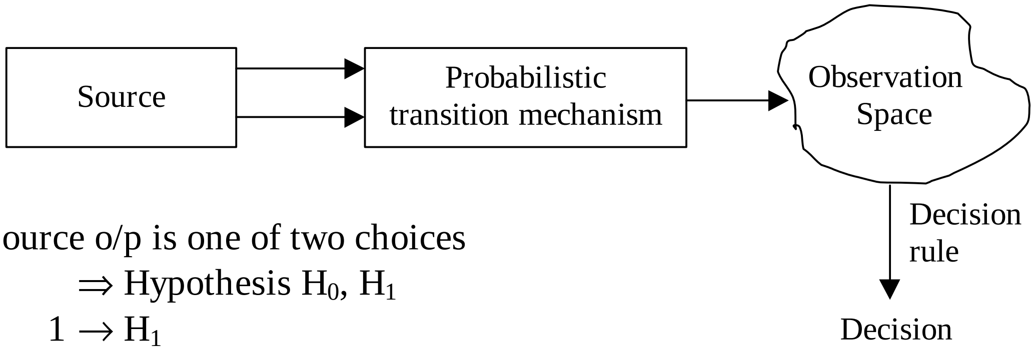
Classical Detection and Estimation Theroy

Reference: "Detection, Estimation and Modulation Theory" by H.L. Van Trees



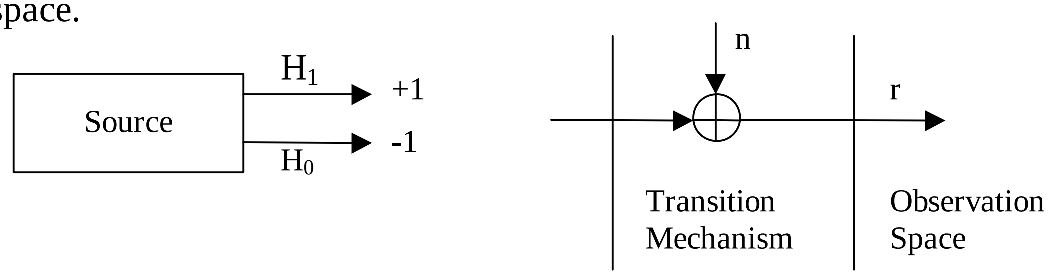
Source o/p is one of two choices

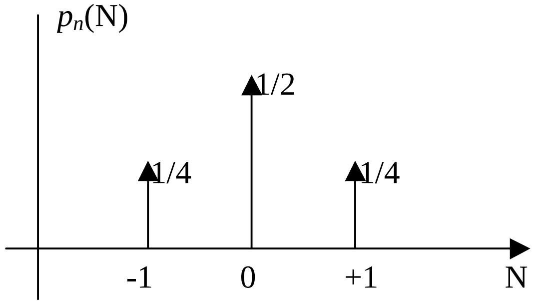
$\Rightarrow$ Hypothesis H_0, H_1

$1 \rightarrow H_1$

$0 \rightarrow H_0$

* Observation Space is finite dimensional, i.e., observations consist of a set of N numbers and can be represented as point in an N-dimensional space.





H_1 : $r = 1+n$

H_0 : $r = -1+n$

$p_{r|H_1}(R|H_1)$, $p_{r|H_0}(R|H_0)$??

Simple Binary Hypothesis Tests

$$\mathbf{r} \cong \begin{bmatrix} r_1 \\ r_2 \\ . \\ . \\ r_N \end{bmatrix}$$

* two known conditional probability densities $p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)$, $p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)$??

$\Rightarrow$ Use this info. to develop a suitable decision rule.

Decision Criteria : Either H_0 or H_1 is true.
One of the following can happen;

1. H_0 true : Choose $H_0 \rightarrow$ correct
2. H_0 true : Choose H_1
3. H_1 true : Choose $H_1 \rightarrow$ correct
4. H_1 true : Choose H_0

Bayes Criterion
Based on

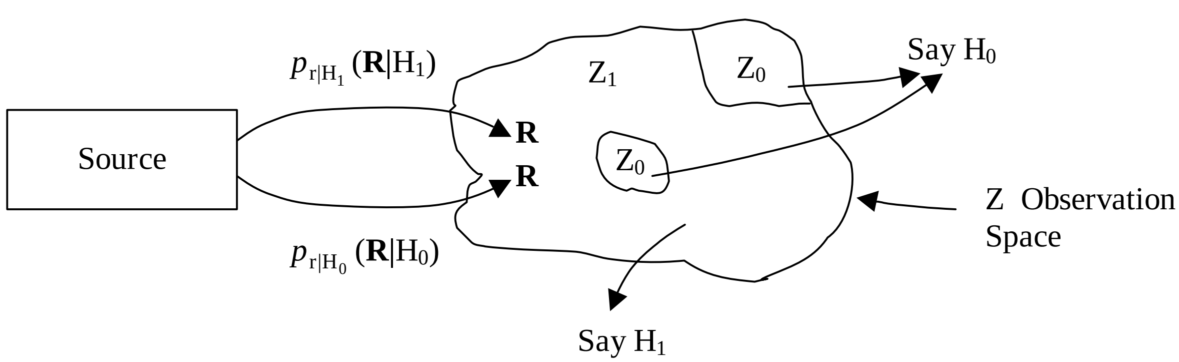
1) The source o/p's are with P_0, P_1 a priori probabilities.
2) A cost is assigned to each possible course of action. C_{00}, C_{10} , C_{11}, C_{01} where $C_{i,\,j} \rightarrow$ the hypothesis true and i is hypothesis chosen.

Each time a certain cost is incurred.
Design decision rule $\Rightarrow$ s.t. "on the average" the cost will be as small as possible

Thus, the expected value of the cost $\Rightarrow$ Risk

$$\begin{aligned} \mathcal{R} &= C_{00}P_0\Pr(\text{say } H_0|H_0 \text{ true}) \\ &+ C_{10}P_0\Pr(\text{say } H_1|H_0 \text{ true}) \\ &+ C_{11}P_1\Pr(\text{say } H_1|H_1 \text{ true}) \\ &+ C_{01}P_1\Pr(\text{say } H_0|H_1 \text{ true}) \end{aligned}$$

Because the decision rule must say either H_1 or H_0 $\Rightarrow$ rule for dividing the total observation space Z into two parts Z_0 and Z_1.



The Risk in terms of transition probabilities and decision region.

$$
\begin{aligned}
\mathsf{R} \quad &= C_{00}P_0 \int_{Z_0} p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\, d\mathbf{R} \\
&+ C_{10}P_0 \int_{Z_1} p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\, d\mathbf{R} \\
&+ C_{11}P_1 \int_{Z_1} p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)\, d\mathbf{R} \\
&+ C_{01}P_1 \int_{Z_0} p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)\, d\mathbf{R}
\end{aligned}
$$

N dimensional space $\Rightarrow$ Integrals are N fold.

<u>Assumption</u> : Cost of a wrong decision is higher than the cost of a correct decision.

$$
\Rightarrow \quad \begin{aligned} C_{10} &> C_{00} \\ C_{01} &> C_{11} \end{aligned}
$$

We must choose regions Z_0 and Z_1 s.t. risk $\Rightarrow$ minimum

$$Z = Z_0 + Z_1$$

Thus

$$
\begin{aligned}
\mathsf{R} \quad &= P_0C_{00} \int_{Z_0} p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\, d\mathbf{R} + P_0C_{10} \int_{Z-Z_0} p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\, d\mathbf{R} \\
&+ P_1C_{01} \int_{Z_0} p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)\, d\mathbf{R} + P_1C_{11} \int_{Z-Z_0} p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)\, d\mathbf{R}
\end{aligned}
$$

$$\int_{Z} p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\, d\mathbf{R} = \int_{Z} p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)\, d\mathbf{R} = 1$$

$$
\begin{aligned}
\Rightarrow \mathsf{R} = P_0C_{10} + P_1C_{11} + \int_{Z_0} \big[&\{P_1(C_{01}-C_{11})\; p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)\} \\
&- \{P_0(C_{10}-C_{00})\; p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\} \big]\, d\mathbf{R}
\end{aligned}
$$

First two terms $\Rightarrow$ fixed cost.

$C_{01}-C_{11} > 0$; $C_{10}-C_{00} > 0$ (assumption)

<u>Thus</u> : all values of **R** where the second term is larger than the first should be <u>included in Z_0</u> because they contribute a negative amount to the integral.

Similarly, all values of **R** where the first term is larger than the second should be excluded in Z_0 (assigned to Z_1) because they contribute a positive amount to the integral.

Thus if:

$$P_1(C_{01}-C_{11})\; p_{\mathbf{r}|H_1}(\mathbf{R}|H_1) \geq P_0(C_{10}-C_{00})\; p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)$$

assign **R** to $Z_1 \Rightarrow H_1$ true ; otherwise assign **R** to $Z_0 \Rightarrow H_0$ true.

$$\Rightarrow \quad \underbrace{\frac{p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)}{p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)}}_{\text{"Likelihood ratio } \Lambda(\mathbf{R})\text{"}} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P_0(C_{10}-C_{00})}{P_1(C_{01}-C_{11})}$$

$$\Lambda(\mathbf{R}) \cong \frac{p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)}{p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)}$$

Because it is the ratio of two functions of a random variable, it is a random variable. We see that regardless of the dimensionality of **R**, $\Lambda(\mathbf{R})$ is a one-dimensional variable.

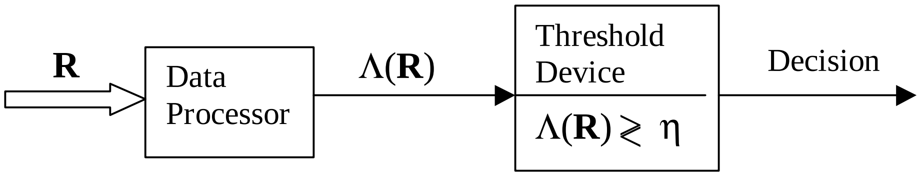
Threshold $\eta \cong \dfrac{P_0(C_{10}-C_{00})}{P_1(C_{01}-C_{11})}$

Thus Bayes Criterion leads us to a "likelihood ratio test".

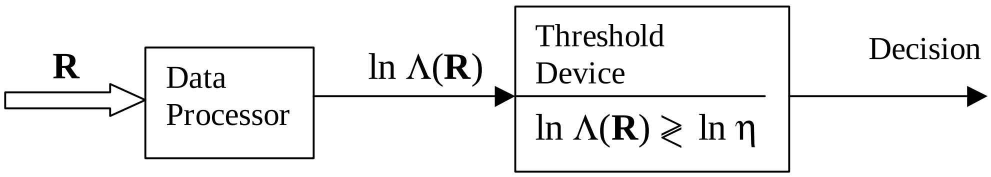
* All the data processing is involved in computing $\Lambda(\mathbf{R})$ and is not affected by a priori probabilities or cost assignments.

Because the natural logarithm is a monotonic function, and equivalent test is

$$\ln \Lambda(\mathbf{R}) \underset{H_0}{\overset{H_1}{\gtrless}} \ln \eta$$



(a)



(b)

Likelihood ratio processors

Example 1

N-observations

Under H_0 : $R_1 = -1 + n_1$ $\quad$ Under H_1 : $R_1 = 1 + n_1$

$R_2 = -1 + n_2$ $\quad$ $R_2 = 1 + n_2$

. $\quad$.

. $\quad$.

$R_N = -1 + n_N$ $\quad$ $R_N = 1 + n_N$

n_i - zero mean, statistically independent Gaussian noise samples of variance σ^2.

$$\Lambda(\mathbf{R}) = \frac{p_{\mathbf{r}|H_1}(\mathbf{R} \mid H_1)}{p_{\mathbf{r}|H_0}(\mathbf{R} \mid H_0)} = \frac{\prod_{i=1}^{N} p_{r_i|H_1}(R_i \mid H_1)}{\prod_{i=1}^{N} p_{r_i|H_0}(R_i \mid H_0)}$$

$$= \frac{\prod_{i=1}^{N} \frac{1}{\sqrt{2ps}} \exp\left(-\frac{(R_i - 1)^2}{2s^2}\right)}{\prod_{i=1}^{N} \frac{1}{\sqrt{2ps}} \exp\left(-\frac{(R_i + 1)^2}{2s^2}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\Rightarrow \therefore \underbrace{\frac{1}{N}\sum_{i=1}^{N} R_i}_{} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{s^2}{2N}\ln h = a$$

$l(\mathbf{R})$ or $l \Rightarrow$ function of the received data

* Sufficient statistics

When making a decision, knowing the value of the sufficient statistic is just as good as knowing $\mathbf{R}$.

* Several kinds of Bayes cost

$\Rightarrow C_{00} = C_{11} = 0$; $C_{01} = C_{10} = 1$

$$\mathsf{R} = P_0 \int_{Z_1} p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\, d\mathbf{R} + P_1 \int_{Z_0} p_{\mathbf{r}|H_1}(\mathbf{R}|H_1) d\mathbf{R}$$

$\Rightarrow$ This is the total probability of making an error.

Thus, for this cost assignment Bayes cost is minimizing the total probability of error. The test is then

$$\Lambda(\mathbf{R}) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P_1}{P_0} \qquad \Rightarrow \underline{\text{minimum error probability}}$$

If in addition $P_0 = P_1$ then

$$\Lambda(\mathbf{R}) = \frac{p_{\mathbf{r}|H_1}(\mathbf{R} \mid H_1)}{p_{\mathbf{r}|H_0}(\mathbf{R} \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1 \Rightarrow \underline{\text{maximum likelihood test}}.$$

We denote the integrals in the Bayes test in the following manner.

$$P_F = \int_{Z_1} p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\, d\mathbf{R}$$

-false alarm : (We say the target is present when it is not.)

From now onward, we'll use the notation $p(X)$ instead of $p_x(X)$.

$$P_D = \int_{Z_1} p(\mathbf{R}|H_1)\, d\mathbf{R}$$

-Detection : (We say the target is present when it is.)

$$P_M = \int_{Z_0} p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)d\mathbf{R} = 1 - P_D$$

-Miss : (We say the target is absent when it is present.)

Consider the earlier example

$$\text{test} \Rightarrow \underbrace{\frac{1}{N}\sum_{i=1}^{N} R_i}_{l} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma^2}{2N}\ln\eta = \alpha$$

- Sum of N statistically independent Gaussian random variables.

$\Rightarrow$ l is Gaussian.

$E[l|H_0] = -1$; $E[l|H_1] = 1$, $\mathrm{var}[l|H_0] = \mathrm{var}[l|H_1] = \dfrac{\sigma^2}{N} = \sigma_1^2$

$$p(l|H_1) = \frac{1}{\sqrt{2\pi}\sigma_1}\exp\left(-\frac{(l-1)^2}{2\sigma_1^2}\right)$$

$$p(l|H_0) = \frac{1}{\sqrt{2\pi}\sigma_1}\exp\left(-\frac{(l+1)^2}{2\sigma_1^2}\right)$$

$\therefore$ $P_F = \Pr[H_1 \text{ chosen} \mid H_0 \text{ true}] = \Pr[l > \alpha \mid H_0 \text{ true}]$

$$= \frac{1}{\sqrt{2\pi}\sigma_1}\int_{\alpha}^{\infty}\exp\left(-\frac{(l+1)^2}{2\sigma_1^2}\right)dl$$

$$= Q\left(\frac{\alpha+1}{\sigma_1}\right)$$

[because $Q(x) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{x}^{\infty}\exp\left(-\frac{x^2}{2}\right)dx$

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{\alpha}^{\infty}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)dy$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\frac{(y-\mu)}{\sigma}}^{\infty}\exp\left(-\frac{z^2}{2}\right)dz \quad \text{(Substituting } \frac{y-\mu}{\sigma} = z\text{)}$$

$$= Q\left(\frac{\alpha-\mu}{\sigma}\right)$$]

$$
\begin{aligned}
P_M &= \Pr[H_0 \text{ chosen} \mid H_1 \text{ true}] = \Pr[l<\alpha \mid H_1 \text{ true}] \\
&= \frac{1}{\sqrt{2\pi}\,\sigma_1} \int_{-\infty}^{\alpha} p(l \mid H_1)\,dl \\
&= 1 - \frac{1}{\sqrt{2\pi}\,\sigma_1} \int_{\alpha}^{\infty} \exp\left(-\frac{(l-1)^2}{2\sigma_1^2}\right) dl \\
&= 1 - Q\left(\frac{\alpha-1}{\sigma_1}\right)
\end{aligned}
$$

Example 2

$H_0 : R_i \rightarrow N(0, \sigma_0^2)$

$H_1 : R_i \rightarrow N(0, \sigma_1^2)$, $\sigma_1 > \sigma_0$, i=1, 2, ..., N

log LRT

$$
\Rightarrow \quad \frac{1}{N}\sum_{i=1}^{N} R_i^2 \underset{H_0}{\overset{H_1}{\gtrless}} T
$$

* For performance need to find statistics of the sufficient statistic $\sum_i R_i^2$ (Chi-squared random variable.)

Minimax Criterion - Useful when a priori probabilities are unknown.

- Use P_F, P_M and P_D.

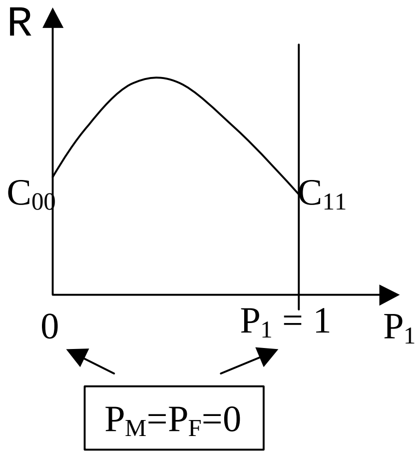
$\Rightarrow$ Bayes Risk

$\mathcal{R} = P_0C_{10} + P_1C_{11} + P_1(C_{01}-C_{11})\,P_M - P_0(C_{10}-C_{00})\,(1-P_F)$

But $P_0 + P_1 = 1 \Rightarrow P_0 = 1- P_1$

$$
\begin{aligned}
\mathcal{R} &= C_{00}(1-P_F) + C_{10}P_F + P_1[(C_{11}-C_{00}) + (C_{01}-C_{11})\,P_M - (C_{10}-C_{00})\,P_F] \\
&= \mathcal{R}_B(P_1) \qquad \text{(because } P_F \text{ and } P_M \text{ are functions of } P_1\text{)}
\end{aligned}
$$

Now, if all costs and a priori probabilities are known, we can find a Bayes test.



* Observe that as P_1 changes the decision regions for Bayes test change and therefore P_F and P_M change.

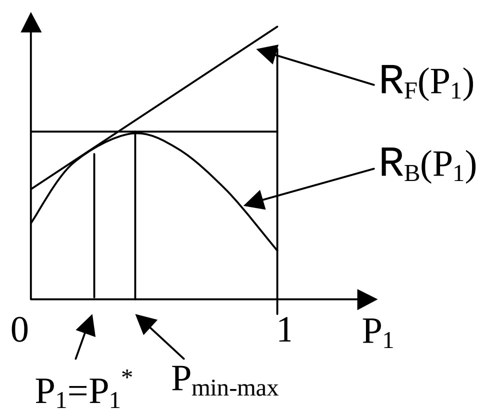
If we assume a P_1, say $P_1 = P_1^*$ and design Bayes test then,

$$\text{Threshold } \eta = \frac{(1-P_1^*)(C_{10}-C_{00})}{P_1^*(C_{01}-C_{11})} \quad \text{is fixed.}$$

and $P_F = \Pr[\Lambda(\mathbf{R})>\eta \mid H_0]$

$P_M = \Pr[\Lambda(\mathbf{R})<\eta \mid H_1]$ are also fixed since η is fixed.

If now P_1 is allowed to vary in the earlier equation $\mathcal{R}_F(P_1)$ is a straight line which touch $\mathcal{R}_B(P_1)$ at $P_1 = P_1^*$.



Thus $\mathcal{R}_F(P_1) \geq \mathcal{R}_B(P_1)$ because the Bayes test minimizes the Risk.
A Bayes test designed to minimize the maximum possible risk is called a "minimax test". Hence we choose $\mathcal{R}_F$ to be the horizontal minimax equation.

$\Rightarrow \quad (C_{11}-C_{00}) + (C_{01}-C_{11})P_M - (C_{10}-C_{00})P_F = 0$

Use $C_{00} = C_{11} = 0$ (This guarantees maximum is interior.)

Let $C_{01} = C_M$

$C_{10} = C_F$ the risk is,

$$\mathcal{R}_B = C_F P_F + P_1(C_M P_M - C_F P_F)$$
$$= P_0 C_F P_F + P_1 C_M P_M$$

The minimax equation is

$$C_M P_M = C_F P_F$$

Neyman - Pearson Criterion

(tries to get around cost-specifications)

$\Rightarrow$ Try to work with conditional probabilities P_D, P_F i.e. make P_F as small as possible, P_D as large as possible.

$\Rightarrow$ Constrain one, and maximize (minimize) the other.

Thus, constrain $P_F = \alpha' \leq \alpha$

Design a test to maximize P_D (<u>or minimize P_M</u>) under this constraint.

Solution to this by "Lagrange mutipliers".

Construct function F;

$$F = P_M + \lambda(P_F - \alpha')$$

$$\text{or} \quad = \int_{Z_0} p(\mathbf{R}|H_1)d\mathbf{R} + \lambda \left[\int_{Z_1} p(\mathbf{R}|H_0)\, d\mathbf{R} - \alpha' \right]$$

$$F = \lambda(1 - \alpha') + \int_{Z_0} [\; p(\mathbf{R}|H_1) - \lambda p(\mathbf{R}|H_0) \;]\, d\mathbf{R}$$

F is minimized by assigning $\mathbf{R} \in Z_0$ whenever $p(\mathbf{R}|H_1) - \lambda\, p(\mathbf{R}|H_0) < 0$.

<u>If $\lambda>0$</u>, then $\Lambda(\mathbf{R}) = \dfrac{p(\mathbf{R} \mid H_1)}{p(\mathbf{R} \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$

To satisfy the constraint λ so that $P_F = \alpha'$.

If the density of $\Lambda \rightarrow p(\Lambda|H_0)$ then under H_0

$$P_F = \int_{l}^{\infty} p(\Lambda|H_0)\, d\Lambda = \alpha' \qquad (**)$$

Solving (**) for λ gives the threshold. The value of λ given by (**) will be non-negative because $p(\Lambda|H_0)$ is zero for negative values of λ.

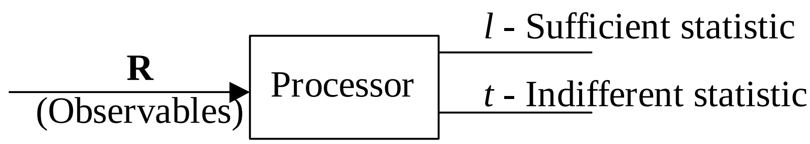
$\Rightarrow$ decreasing λ is equivalent to increasing $Z_1 \Rightarrow H_1$.

Therefore we decrease λ until we obtain the largest possible $\alpha' \leq \alpha$.

$\Rightarrow$ In most cases P_F is a continuous function of λ and we have $P_F = \alpha$.

Under this assumption the Neyman-Pearson criterion leads to a likelihood ratio test (LRT).

<u>Sufficient Statistic</u>



t is indifferent if $p(t|H_1, l) = p(t|H_0, l)$

Since $\Lambda(l, t) = \dfrac{p(l,t \mid H_1)}{p(l,t \mid H_1)} = \dfrac{p(l \mid H_1)\,p(t \mid H_1, l)}{p(l \mid H_0)\,p(t \mid H_0, l)}$

$$= \frac{p(l \mid H_1)}{p(l \mid H_1)}$$

$\Rightarrow$ knowledge of only l allows one to distinguish between hypothesis.

<u>e.g</u>

i) $H_1 : R_1 = m + n_1$ $\quad\quad$ $H_0 : R_1 = n_1$

$\quad R_2 = n_2$ $\quad\quad\quad\quad\quad\quad$ $R_2 = n_2$

n_1 , n_2 are statistically independent. R_2 – irrelevant statistics

Note i) If l is passed through an invertible operation $\Rightarrow$ results in a sufficient statistic.

ii) (l, t) together are sufficient statistics through <u>not</u> minimal sufficient statistics.

<u>Performance : Receiver Operating Characteristic</u>

Evaluating the perfromance of the LRT.

Neyman – Pearson test $\Rightarrow P_F$, P_D specify the performance.

Bayes Risk R_B follows from P_F, P_D.

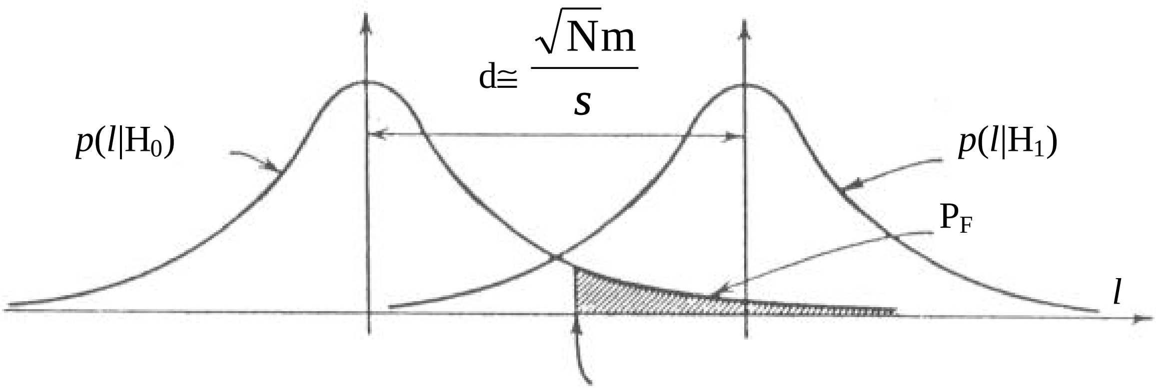
$\Rightarrow$ Concentrate on calculating P_F and P_D.

e.g

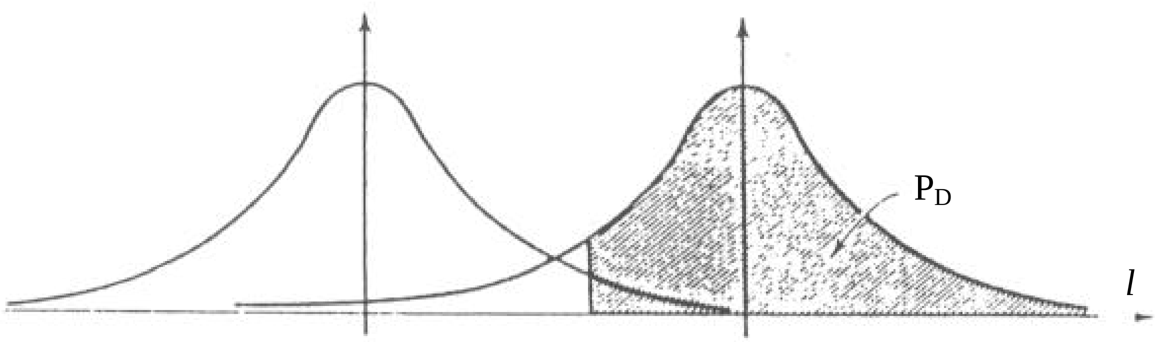
$H_1 : r_i = m + n_i \quad , \quad i = 1, 2, \ldots, N$

$H_0 : r_i = n_i \quad , \quad i = 1, 2, \ldots, N$

$$\Rightarrow l = \frac{1}{\sqrt{N}\, s} \sum_{i=1}^{N} R_i \underset{H_0}{\overset{H_1}{\gtrless}} \frac{s}{\sqrt{N} m} \ln h + \frac{\sqrt{N} m}{2 s}$$



$$\text{Threshold : } \frac{s}{\sqrt{N}m} \ln h + \frac{\sqrt{N}m}{2s} = \frac{\ln h}{d} + \frac{d}{2}$$



$l \sim N(0, 1)$ under H_0.

$\sim N(\frac{\sqrt{N}m}{s}, 1)$ under H_1.

$d \cong \frac{\sqrt{N}m}{s}$ is the distance between the means of two densities.

Thus,

$$P_F = \int_{\frac{\ln h}{d}+\frac{d}{2}}^{\infty} \frac{1}{\sqrt{2p}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$= Q\left(\frac{\ln h}{d} + \frac{d}{2}\right)$$

$$P_D = \int_{\frac{\ln h}{d}+\frac{d}{2}}^{\infty} \frac{1}{\sqrt{2p}} \exp\left(-\frac{(x - d)^2}{2}\right) dx$$

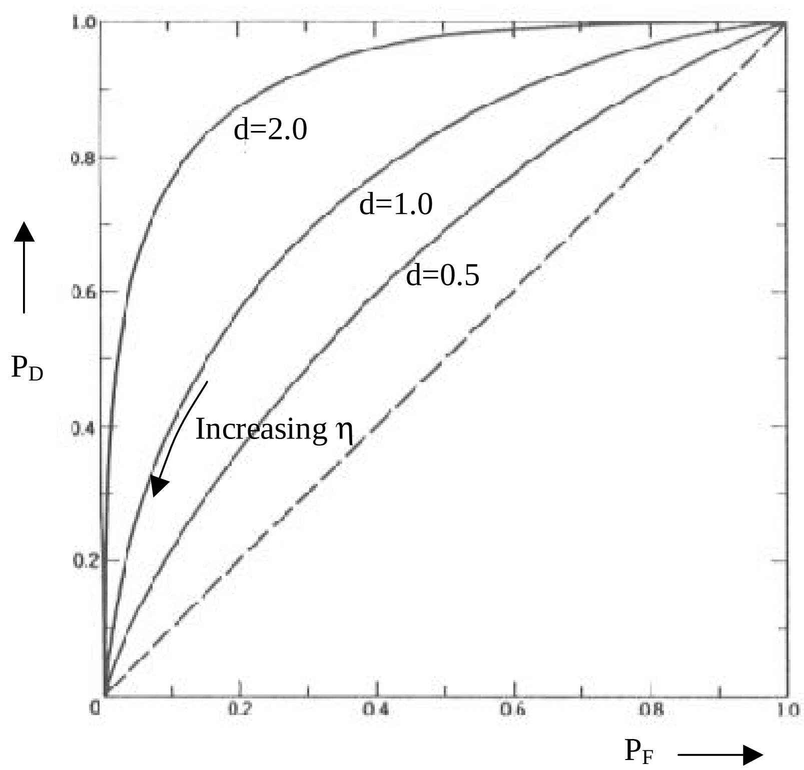
$$= \int_{\frac{\ln h}{d}-\frac{d}{2}}^{\infty} \frac{1}{\sqrt{2p}} \exp\left(-\frac{y^2}{2}\right) dy$$

$$= Q\left(\frac{\ln h}{d} - \frac{d}{2}\right)$$

Plot P_D versus P_F for various values of d with η as the varying parameter.

$\eta = 0$; $\ln \eta = -\infty \Rightarrow P_F = P_D = 1\ (H_1)$

$\eta \to \infty$; $P_F = P_D = 0\ (H_0)$



Receiver Operating Characteristic (ROC)

Performance increases monotically with d. (As we would expect)

<u>Special Case</u> : Minimizing the total probability of error.

$$\Pr(\in) \cong P_0 P_F + P_1 P_M$$
$$P_0 = P_1 = ½ \ (\eta = 1)$$
$$\Pr(\in) = ½ \ (P_F + P_D)$$
$$= \int_{\frac{d}{2}}^{\infty} \frac{1}{\sqrt{2p}} \exp\left(-\frac{x^2}{2}\right) dx$$
$$= Q\left(\frac{d}{2}\right)$$

Bounds for Q(x) : For x > 0

$$\frac{1}{\sqrt{2p}\,x}\left(1 - \frac{1}{x^2}\right)\exp\left(-\frac{x^2}{2}\right) < Q(x) < \frac{1}{\sqrt{2p}\,x}\exp\left(-\frac{x^2}{2}\right)$$

$$Q(x) < \frac{1}{2}\exp\left(-\frac{x^2}{2}\right) \qquad ; \qquad x>0$$

<u>Example 2</u>
Unequal variances

$$l(\mathbf{R}) = \sum_{i=1}^{N} R_i^2 \quad \underset{H_0}{\overset{H_1}{\gtrless}} \quad \frac{2s_0^2 s_1^2}{s_1^2 - s_0^2}\left(\ln h \text{ - } N \ln \frac{s_0}{s_1}\right)$$

$$= \gamma \ (\sigma_1 > \sigma_0)$$

<u>Consider N=2</u>

$P_F = \Pr(l \geq \gamma | H_0)$
$\quad = \Pr(r_1^2 + r_2^2 \geq \gamma | H_0)$

r_1, r_2 statistically independent var. σ_0^2
$r_1 = z \cos\theta$ and $r_2 = z \sin\theta$
$z = \sqrt{r_1^2 + r_2^2}$ and $\theta = \tan^{-1}(r_2/r_1)$

$$\Pr(z^2 \geq \gamma|H_0) = \int_0^{2\pi} d\theta \int_{\sqrt{\gamma}}^{\infty} z \frac{1}{2\pi\sigma_0^2} \exp\left(-\frac{z^2}{2\sigma_0^2}\right) dz$$

$$= \int_{\sqrt{\gamma}}^{\infty} \frac{1}{\sigma_0^2} .z \exp\left(-\frac{z^2}{2\sigma_0^2}\right) dz$$

$$= \int_{\gamma}^{\infty} \frac{1}{2\sigma_0^2} \exp\left(-\frac{\lambda}{2\sigma_0^2}\right) d\lambda = \exp\left(-\frac{\gamma}{2\sigma_0^2}\right)$$

Similarly, $P_D = \exp\left(-\frac{\gamma}{2\sigma_1^2}\right)$

$$P_D = (P_F)^{\sigma_0^2/\sigma_1^2} \quad \text{(ROC)}$$

$$\ln P_D = \frac{\sigma_0^2}{\sigma_1^2} \ln P_F$$

$P_F < 1 \quad \Rightarrow \quad P_D \uparrow \qquad \sigma_1^2/\sigma_0^2 \uparrow$

e.g

$\sigma_1^2/\sigma_0^2 = 2 \qquad P_F = 0.2 \qquad P_D = (0.2)^{1/2} = 0.447$

$\sigma_1^2/\sigma_0^2 = 4 \qquad P_D = (0.2)^{1/4} = 0.6687$

Example 3

Poisson distribution of events : Our observation is just this number which obeys a Poisson distribution on both hypothesis, i.e.,

$$\Pr(\text{ n events }) = \frac{(m_i)^n e^{-m_i}}{n!}, \qquad n = 0, 1, 2, \ldots$$
$$i = 0, 1$$

$$E[n] = m_i$$

$$\Lambda(n) = \left(\frac{m_1}{m_0}\right)^n \exp[-(m_1 - m_0)] \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$n \ln \frac{m_1}{m_0} - (m_1 - m_0) \underset{H_0}{\overset{H_1}{\gtrless}} \ln \eta$$

$$n\,[\,\ln m_1 - \ln m_0]\ \underset{H_0}{\overset{H_1}{\gtrless}}\ \ln \eta + (\,m_1 - m_0)$$

If $m_1 > m_0$,

$$n\ \underset{H_0}{\overset{H_1}{\gtrless}}\ \frac{\ln h + (m_1 - m_0)}{\ln m_1 - \ln m_0}$$

If $m_1 < m_0$,

$$n\ \underset{H_1}{\overset{H_0}{\gtrless}}\ \frac{\ln h + (m_1 - m_0)}{\ln m_1 - \ln m_0}$$

Now consider the case $m_1 > m_0$.

$$n\ \underset{H_0}{\overset{H_1}{\gtrless}}\ \frac{\ln h + (m_1 - m_0)}{\ln m_1 - \ln m_0} = \gamma_1 \qquad \text{consider the integer values}$$

integer only

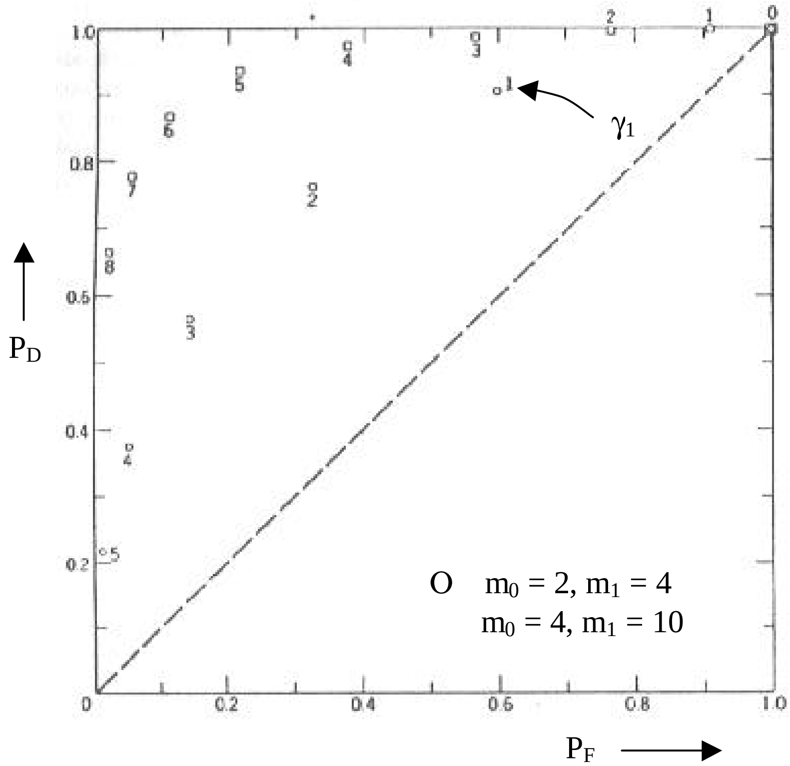
$$\Rightarrow \qquad n\ \underset{H_0}{\overset{H_1}{\gtrless}}\ \gamma_1 \quad , \gamma_1 = 0, 1, 2, \ldots$$

Now,

$$\begin{aligned} P_D &= \Pr[n \geq \gamma_1 | H_1] = 1 - \Pr[n < \gamma_1 | H_1] \\ &= 1 - \sum_{n=0}^{g-1} \frac{(m_1)^n e^{-m_1}}{n!} \qquad , \gamma_1 = 0, 1, 2, \ldots \end{aligned}$$

Similarly,

$$\begin{aligned} P_F &= \Pr[n \geq \gamma_1 | H_0] = 1 - \Pr[n < \gamma_1 | H_0] \\ &= 1 - \sum_{n=0}^{g-1} \frac{(m_0)^n e^{-m_0}}{n!} \end{aligned}$$



<u>ROC</u>
* consists of a series of points
$\gamma_1 \rightarrow 0 - 1$
$P_F \rightarrow 0 \rightarrow 1\text{-}e^{-m_0}$

LRT	γ_1	P_F	P_D
0	0	1	1
1	1	$1\text{-}e^{-m_0}$	$1\text{-}e^{-m_1}$

If P_F to have an intermediate value between 1 and $1\text{-}e^{-m_0}$, say $1\text{-}\frac{1}{2}e^{-m_0}$.

$$\underset{LRT0}{P_F} \qquad \underset{LRT1}{P_F} \qquad \Rightarrow \qquad \frac{1}{2}.\underset{LRT0}{P_F} + \frac{1}{2}\underset{LRT1}{P_F}$$

$$= \frac{1}{2}.1 + \frac{1}{2}.(1\text{-}e^{-m_0})$$

$$= 1\text{-}\frac{1}{2}.e^{-m_0}$$

Therefore the test is

If $n = 0$, say H_1 with probability $\frac{1}{2}$.

say H_0 with probability $\frac{1}{2}$.

$n \geq 1$, say H_1

Here we mix two LRT's in same probabilistic manner

$\Rightarrow$ <u>randomized decision rule</u>

$$P_D = \frac{1}{2} \cdot \underset{\text{LRT0}}{P_D} + \frac{1}{2} \cdot \underset{\text{LRT1}}{P_D}$$

$$= 0.5 \cdot 1 + 0.5\,(1 - e^{-m_1})$$

$$= 1 - \frac{1}{2} \cdot e^{-m_1}$$

<u>Reason</u> : Observed r.v's are discrete $\Rightarrow \Lambda(\mathbf{R})$ is a discrete r.v.
Only certain values of P_F possible.