Representations for TCM

Ungerboeck representation: (considered earlier)

- A binary convolutional encoder
- A modulator part.

Analytical representation:

The effect of the trellis encoder is to transform a sequence of input symbols to a sequence of modulated- channel signals.

$x_{\mathrm{I}} = f(a_I, a_{I-1},\ldots\ldots\ldots, a_{I-L})$

The function *f(.)* must be *nonlinear*.

Expanding the function *f* :

Let $a_1, a_2,\ldots\ldots, a_n$ take on values in the set {0,1}.

$$f(a_1, a_2,\ldots\ldots\ldots, a_n) = k^{(0)} + S(i)\, k_I^{(1)}\, a_I + S\,(j>i)\, k_{ij}^{(2)}\, a_I a_j + S\,(h>j>i)\, k_{hji}^{(3)}\, a_i a_j a_h + \ldots\ldots\ldots + k_{12\ldots n}^{(n)}\, a_1 a_2 \ldots\ldots a_n$$

where the coefficients $k^{(l)}$ are a set of constants that may be determined from the 2^n values the function *f* can assume.

In TCM it is easier to deal with {-1,+1} than {0,1} initially.

$$b_I = 1 - 2\, a_I$$

$$f(b_1, b_2,\ldots\ldots\ldots, b_n) = d^{(0)} + S(i)\, d_I^{(1)}\, b_I + S\,(j>i)\, d_{ij}^{(2)}\, b_I b_j +$$

$$S\,(h>j>i)\, d_{hji}^{(3)}\, b_i b_j b_h + \ldots\ldots\ldots + d_{12\ldots n}^{(n)}\, b_1 b_2 \ldots\ldots b_n \quad \text{---------(****)}$$

Let **f** denote a column vector whose 2^n components are all the values that f can take.

Let **d** denote the column vector of the (unknown) coefficients.

Finally let **B** be a 2^n x 2^n matrix where each row represents the 2^n values taken by all the products of the b_I's in (****) for **each - specific** *n*-tuple $b_1, b_2, \ldots, b_n$ for the corresponding value of f.

Then (****) can be given in the matrix form

$$\mathbf{f} = \mathbf{B}\,\mathbf{d}$$

We see that **B** has orthogonal rows (and columns), and thus

$$\mathbf{B}\,\mathbf{B}^{\mathrm{T}} = 2^n\,\mathbf{I}$$

B is a hadamard Matrix.

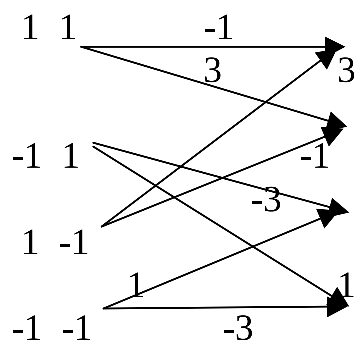
$$\mathbf{B}^{-1} = (1 / 2^n)\,\mathbf{B}^{\mathrm{T}}$$

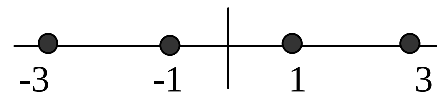
Therefore the vector of the unknown coefficients can be computed easily by

$$\mathbf{d} = (1 / 2^n)\,\mathbf{B}^{\mathrm{T}}\,\mathbf{f}$$

We can say that **d** is the Hadamard transform of the vector **f**.

Example:





4-AM signals

A rate ½ TCM scheme with 4-AM signals with memory v=2.

The values along the transitions between states represent the output symbols $f(b_1,b_2,b_3)$, that is going from present state (b_2, b_3) to the next state (b_1, b_2) in response to the input symbol b_1.

For $n=3$ by choosing $d^{(0)} = 0$

$$f(b_1,b_2,b_3) = d_1^{(1)} b_1 + d_2^{(1)} b_2 + d_3^{(1)} b_3 + d_{12}^{(2)} b_1 b_2 + d_{12}^{(2)} b_1 b_3$$

$$+ d_{12}^{(2)} b_2 b_3 + d_{123}^{(3)} b_1 b_2 b_3$$

From the trellis diagram, we have

$f(1,1,1) = -1$ $\quad$ $f(-1,1,1)=3$ $\quad$ $f(1,-1,1)=-3$ $\quad$ $f(-1,-1,1)=1$
$f(1,1,-1)=3$ $\quad$ $f(-1,1,-1)=-1$ $\quad$ $f(-1,-1,-1)=-3$ $\quad$ $f(1,-1,1)=1$

We have

$$\mathbf{f}^T = [\ -1 \quad 3 \quad -3 \quad 1 \quad 3 \quad -1 \quad -3 \quad 1]$$

$$\mathbf{d}^T = [\ d_1^{(1)} \quad d_2^{(1)} \quad d_3^{(1)} \quad d_{12}^{(2)} \quad d_{13}^{(2)} \quad d_{23}^{(2)} \quad d_{123}^{(3)}\ ]$$

and

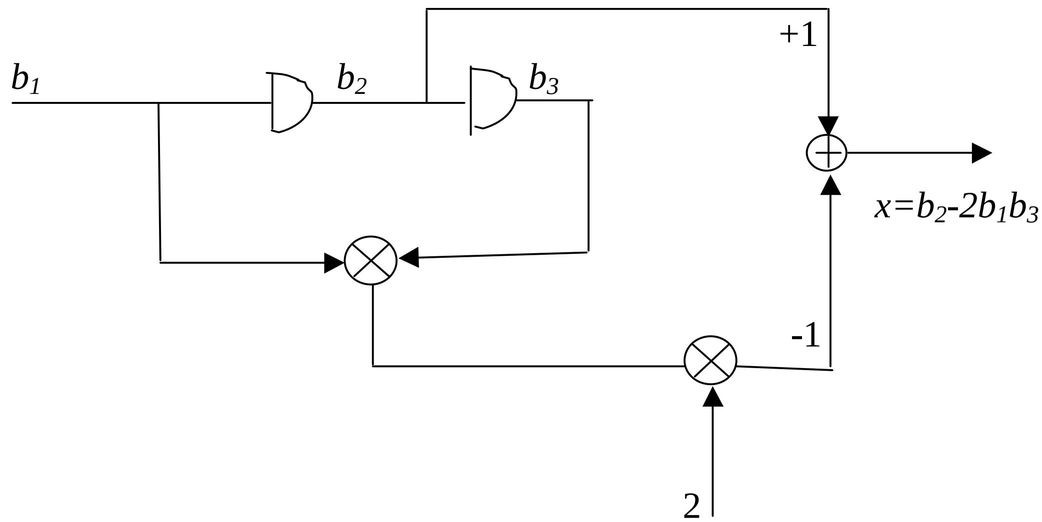
$$B = \begin{bmatrix} f(1=b_1, 1=b_2, 1=b_3) \\ f(-1,1,1) \\ f(1,-1,1) \\ f(-1,-1,1) \\ f(1,1,-1) \\ f(-1,1,-1) \\ f(-1,-1,-1) \\ f(1,-1,1) \end{bmatrix} =$$

$$\begin{bmatrix} 1(b_1) & 1(b_2) & 1(b_3) & 1(b_1 b_2) & 1(b_1 b_3) & 1(b_2 b_3) & 1(b_1 b_2 b_3) \\ -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix}$$

We get

$$f(b_1,b_2,b_3) = b_2 - 2\, b_1\, b_3$$

Implementation: *combined* modulation / coding process



Consider now z_1 and z_0 as representing output MSB and LSB respectively.

Output MSB	z_1	-1	-1	1	1
LSB	z_0	1	-1	1	-1
signal	x	3	1	-1	-3

$x = -2z_1 + z_0$

From above

$x = b_2 - 2\, b_1\, b_3$

by comparing

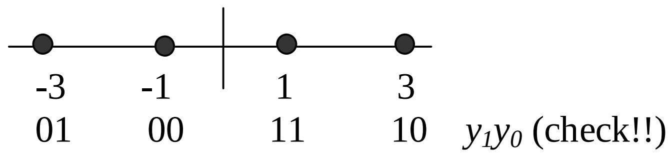
$z_1 = b_1 b_3$ *MSB* $\quad z_0 = b_2$ *LSB*

The corresponding relations for 0 and 1 variables are

$y_1 = a_1 + a_3 \pmod 2$ - *MSB* $\qquad y_0 = a_2$ *LSB*

This will give the convolutional encoder / mapper realization.

$y_1 = a_1 + a_3$ *(mod 2) - MSB* $\quad y_0 = a_2$ *LSB*



Similar analytical expressions can be obtained for PSK, QAM schemes and also for MTCM schemes.

References for this section:

1. R. Calderbank and J. E. Mazo, “ A new description of trellis codes,” IEEE Transactions Information Theory, Vol. IT-30, pp. 784-791, Nov. 1984.
2. TCM Text: Chapter 5

Further Reading:

Multidimensional Modulations
Rotationally Invariant Trellis Codes (RIC)

Channel Coding for Telecommunications: Chapter 10 on Coded Modulation (Martin Bossert)

Performance Analysis of TCM

- The error event of length d is the event that the decoder chooses the incorrect sequence $\hat{x}$ instead of the transmitted sequence x corresponding to the incorrect trellis path that leaves from the correct path and remerges with it exactly d discrete time later.

- The union bound states that the sum of the probabilities of the individual events is greater than or equal to the probability of the union of the events.

- Let us assume that the code is has uniform error property, i.e., the error probability doesn't depend on the transmitted sequence. Ungerboeck's TCM is usually the case.

- The upper bound of the bit error probability is usually obtained through the union bound as

$$P_b = \frac{1}{k}\sum_{d=0}^{\infty} a_d(x,\hat{x}) P_d(x \rightarrow \hat{x}), \qquad (1)$$

where k is the number of bits per encoding interval,

$a_d(x,\hat{x})$ is the number of error bits corresponding to the error event $(x,\hat{x})$ of length d,

$P_d(x \rightarrow \hat{x})$ is the pairwise error probability, i.e., the probability of decoder choosing the sequence $\hat{x}$ instead of the transmitted sequence x.

- From (1) we can assume that the transmitted sequence is all zero sequence. The incorrect path is a path leaving from the all zero sequence and merging with it later.

TCM for Frequency-Flat Fading Channels

- The design criterion of TCM in AWGN channels is to maximize the free Euclidean distance (d^2_{free}) [1].

- However, TCM with optimum (d^2_{free}) might not perform well in frequency-flat fading channels.

- Divsalar and Simon derived the performance bound of TCM for frequency-flat fading channels [2].

- Assumption:
  - Frequency-flat Rician fading channel.
  - Ideal interleaving (uncorrelated fading)
  - Perfect channel estimation

$$P(x \to \hat{x}) \le \prod_{k \in h} \frac{1+K}{1+K+\frac{E_s}{4N_0}|x_k-\hat{x}_k|^2} \exp\left\{-\frac{K\frac{E_s}{4N_0}|x_k-\hat{x}_k|^2}{1+K+\frac{E_s}{4N_0}|x_k-\hat{x}_k|^2}\right\} \quad (2)$$

where K is the ratio of power in the direct plus specular component tothat in the diffuse component,

E_s is the symbol energy,

N_0 is the one-sided noise power spectral density,

h is the set of time k such that $x_k \ne \hat{x}_k$.

- Eq. (2) can be written as

$$P(x \to \hat{x}) \le \exp\left(-\frac{E_s}{4N_0}d^2\right) \quad (3)$$

with

$$d^2 = \sum_{k \in h} \Bigg\{ \underbrace{\frac{|x_k-\hat{x}_k|^2 K}{1+K+\frac{E_s}{4N_0}|x_k-\hat{x}_k|^2}}_{d^2_{1k}} + \underbrace{\left(\frac{E_s}{4N_0}\right)^{-1} \ln\left(\frac{1+K+\frac{E_s}{4N_0}|x_k-\hat{x}_k|^2}{1+K}\right)}_{d^2_{2k}} \Bigg\} \quad (4)$$

- Note that for $K = \infty$ (no fading)

$$d_{1k}^2 = |x_k - \hat{x}_k|^2 \quad \text{and} \quad d_{2k}^2 = 0 \tag{5}$$

and d^2 is merely the sum of the Euclidean distance along the error event path

- For $K = 0$ (Rayleigh fading)

$$d_{1k}^2 = 0 \quad \text{and} \quad d_{2k}^2 = \left(\frac{E_s}{4N_0}\right)^{-1} \ln\left(1 + \frac{E_s}{4N_0}|x_k - \hat{x}_k|^2\right) \tag{6}$$

and for large E_s/N_0 values, the pairwise error probability becomes

$$P(x \to \hat{x}) \le \left(\prod_{k \in h} \frac{E_s}{4N_0}|x_k - \hat{x}_k|^2\right)^{-1} \tag{7}$$

i.e., it is inversely proportional to the product of the squared Euclidean distance along the error event path.

- Substituting (7) into (1), we have

$$P_b \le \frac{1}{k}\sum_{d=0}^{\infty} a(x, \hat{x}) \prod_{k \in h} \frac{1}{\frac{E_s}{4N_0}|x_k - \hat{x}_k|^2} \tag{8}$$

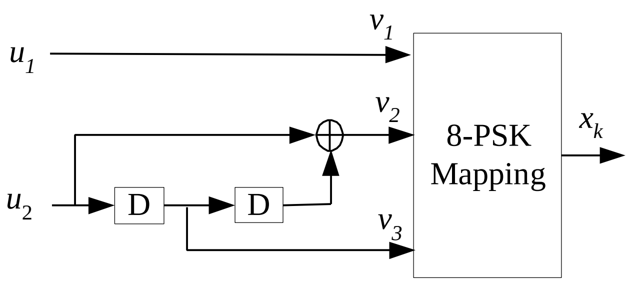
- The bit error probability will be dominated by the term that has the slowest rate of descent with E_s/N_0. This corresponds to the error event path with the smallest number of symbols in h . This path is referred to "shortest error event path".

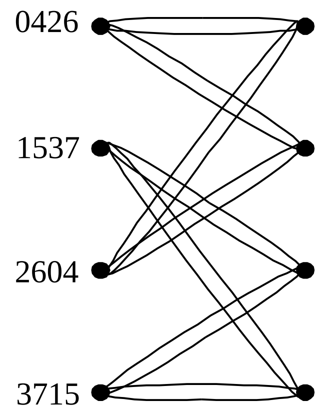
- Define the length of the shortest error event path by the number of nonzero pairwise distance between symbols along the error event.

- Therefore, the design criterion of TCM in the frequency-flat independent fading channel is to maximize the length of the shortest error event path and the product of the branch distance along that path.

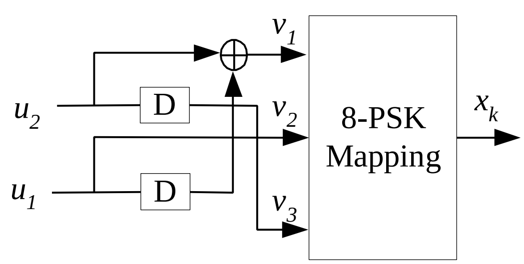
- Note that the length of the shortest error event path is also called effective code length (ECL).

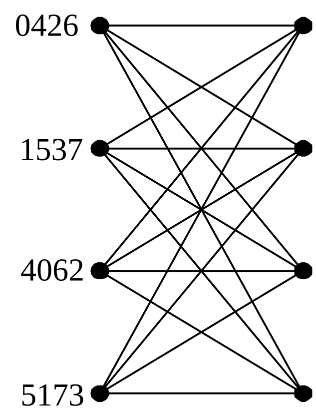
<u>Example</u> TCM (1) Four-state Ungerboeck's TCM [1]
TCM (2) Four-state Wilson/Leung's TCM [3]





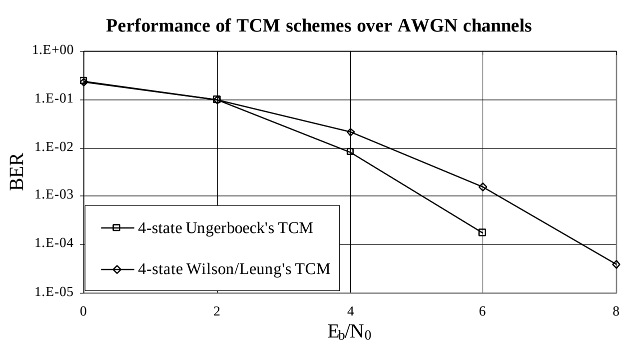
Four-state Ungerboeck's TCM

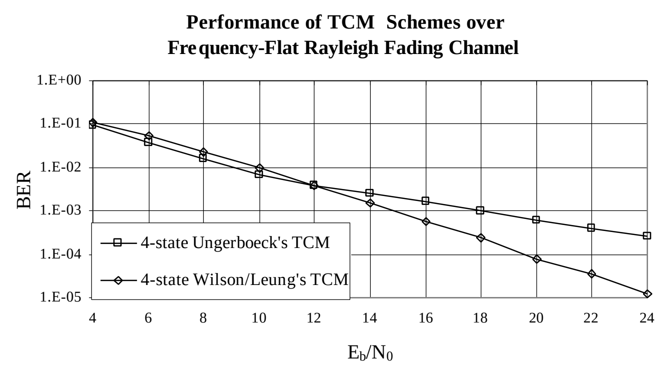




Four-state non-parallel transition TCM

Code	Effective Code Length	Minimum Euclidean distance	Minimum Product distance
TCM (1)	1	4.00	4.00
TCM (2)	2	2.59	1.17





TCM for Frequency-Selective Fading Channels

- In frequency-selective fading channels, the performance of codes deteriorates because there is overlapping between adjacent symbols or ISI. Equalization can mitigate the effect of ISI.

- Joint MLSE equalization and decoding of TCM was discussed by Chiu and Chao [4]. Viterbi decoding is performed on the combined trellis of the channel and decoder.

- The design criteria of TCM in frequency-selective fading channels with joint MLSE equalization and decoder were concluded in [5]. The parameters that affect the performance are the length of the shortest error event path and the product of the ISI distance defined as:

$$\text{ISI distance} = \sum_{j=0}^{L} s_{gj}^{2} \left| x_{k-j} - \hat{x}_{k-j} \right|^{2} \tag{9}$$

where L is the channel memory,

s_{gj}^{2} is half of the variance of the channel impulse response of path jth,

- However, Hamied and Stuber [6] found that uncoded system outperforms the trellis-coded system when joint MLSE equalization and decoding is used. This is because the interleaving cannot be used.

- A more appropriate technique is to use a separate equalizer and a decoder with a deinterleaver between them.

- Various soft-output equalizers have been employed with TCM in [7]. The results showed the benefit of soft-output over hard-output significantly.

- Chen and Chuang [8] studied the performance of TCM without equalization. They concluded that both built-in time diversity (length of the shortest error event path) and Euclidean distance are important for TCM schemes in frequency-selective fading channels.

- The new TCM code design in frequency-selective fading channels remains open for various types of receivers.

References

1. G. Ungerboeck, "Channel Coding with Multilevel/Phase Signals",*IEEE Trans. Inform. Theory*, vol. IT-28, no. 1, pp. 55-67, Jan.1982.
2. D. Divsalar and M.K. Simon, "The Design of Trellis Coded MPSK for Fading Channels: Performance Criteria",*IEEE Trans. Commun*., vol. 36, no. 9, pp. 1004-1012, Sept.1988.
3. S.G. Wilson and Y.S. Leung, "Trellis-Coded Phase Modulation on Rayleigh Channels", *Proceeding of the 1987 Int. Conf. on Commun.(ICC'87)* , pp. 739-743, 1987.
4. M. Chiu and C. Chao, *Performance of Joint Equalization and Trellis-Coded Modulation on Multipath Fading Channels* , IEEE Trans. Commun., vol. 43, no. 2/3/4, pp. 1230-1234, Feb./Mar./Apr. 1995.
5. P. Tarasak and N. Rajatheva, *Upper bound on the bit error probability of TCM over frequency-selective Rayleigh fading channel* , Electronics Letters, vol. 35, no. 21, pp. 1820-1821, Oct 14th, 1999.
6. A. Hamied and G. L. Stuber, *Performance of Trellis Coded Modulation for Equalized Multipath Fading ISI Channels* , IEEE Trans. Vehicular Tech., vol. 44, no. 1, pp. 50-58, Feb. 1995.
7. N.H. Ha and R.M.A.P. Rajatheva, *Performance of Trellis-Coded Modulation on Frequency-Selective Rayleigh Fading Channels with Soft-output Algorithms* , in VTC'98 Conf. Rec., 48th IEEE Vehicular Technology Conference 1998, vol. 1., 1998.