S-72.333 Postgraduate course in radio communications

Channel Capacity, MIMO capacity

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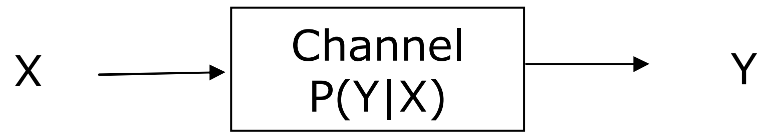
Outline

- Introduction
- Definitions of Channel and Channel Capacity
- Capacity of Discrete Memoryless Channel
- Capacity of Gaussian Channel
- MIMO capacity
- Summary and conclusions

Introduction

- Channel capacity is a fundamental limit on the rate of error free messages that can take place over a channel
- Advantage of Multiple-Input Multiple-Output (MIMO) systems is increased capacity

Definitions (Shannon)



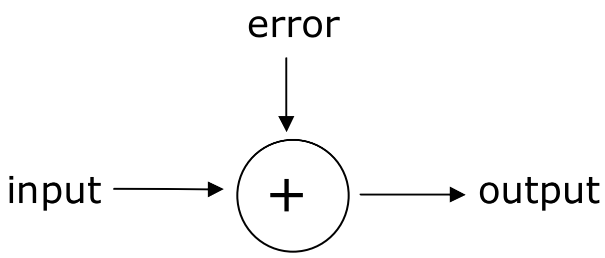
- **Channel is defined by conditional distribution of channel output given input**
  - Channel input and output are random variables where distribution of the input determines distribution of the output
  - Mutual information, I(X;Y), can be considered as amount of the information conveyed through channel. It depends on the input distribution.
- **Channel capacity is maximum mutual information of input and output**

$$C = \max_{p(x)} I(X;Y)$$

Example: Binary Symmetric Channel

In Binary Symmetric Channel (BSC)
output = input + error *mod 2*
where ***input, output*** and ***error*** *are binary random variables taking values 0 or 1.*
Bit error probability is

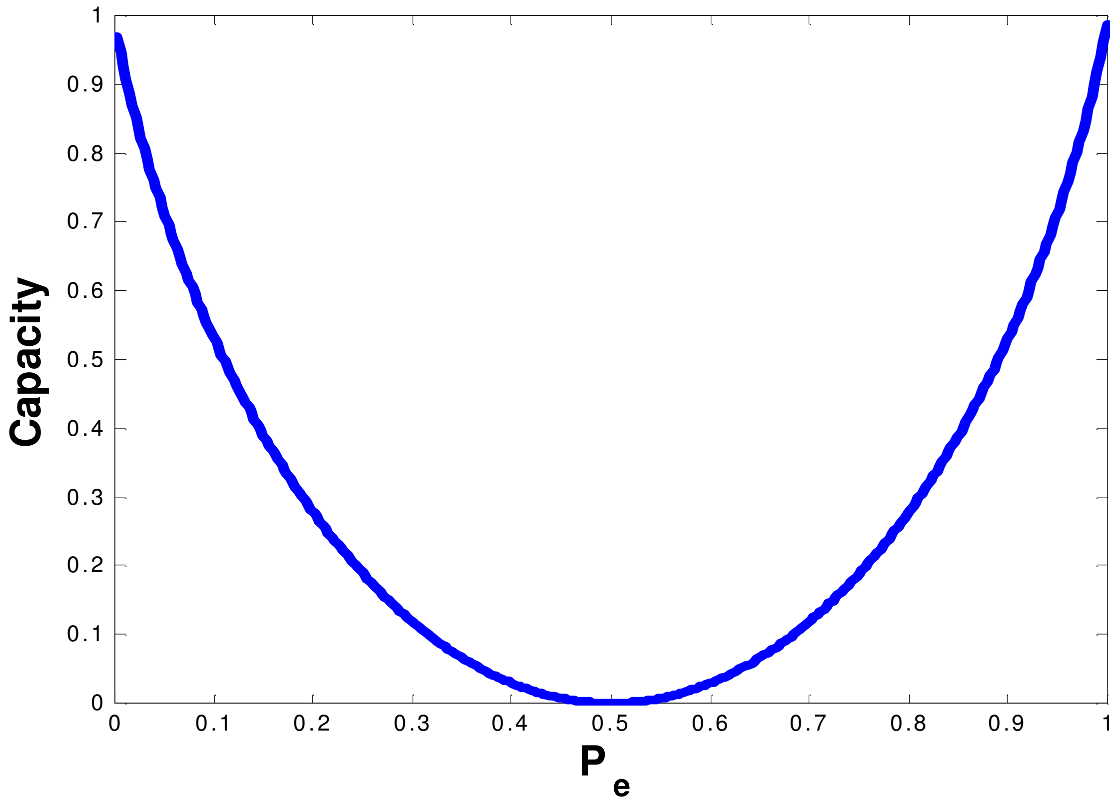
$$P_e = P(error = 1)$$



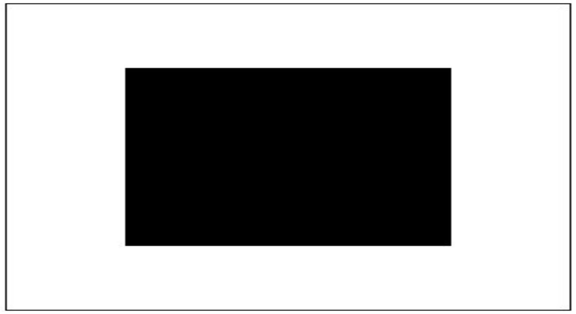
$$P(output \mid input) = \begin{bmatrix} 1 - p_e & p_e \\ p_e & 1 - p_e \end{bmatrix}$$

Example: Binary Symmetric Channel

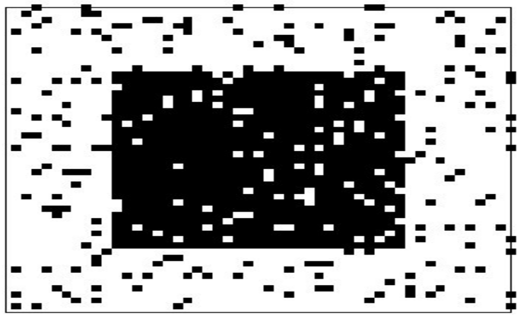
$$C_{BSC} = 1 + p_e \log_2 p_e + (1 - p_e)\log_2(1 - p_e)$$



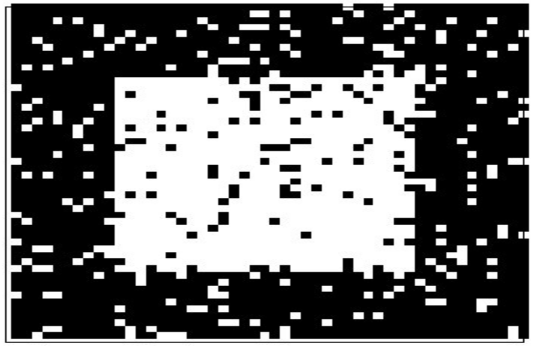
Example: Binary Symmetric Channel



Pe = 0, C = 1



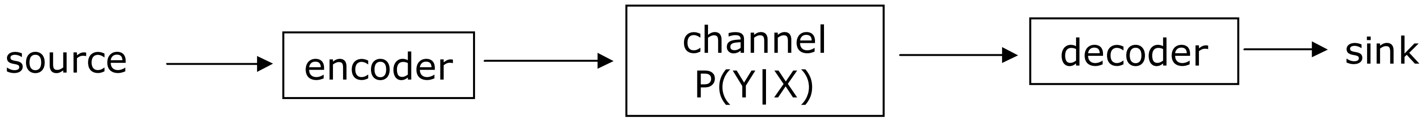
Pe = 0.1, C = 0.5



Pe = 0.9, C = 0.5

Pe = 0.5, C = 0

Channel Coding Theorem



- Channel coding theorem (Shannon 1948) says that reliable information rates up to channel capacity are possible.
- Channel coding theorem do not say how to achieve these reliable rates. Efforts on designing efficient channel codes has been going on ever since.
- Theorem do not promise non-erroneous transmissions when application has delay constraints. Result of the theorem is asymptotic: non-erroneus transmissions require unlimited block lengths.

Capacity of Gaussian Channel

- Gaussian channel is time discrete continuous alphabet channel. Output is sum of the input and a Gaussian noise, where input and noise are independent.

$$Y = X + Z \qquad\qquad Z \sim \mathbf{N}(0, N)$$

- Since input and noise are independent, mutual information of output and input is differential entropy of output minus diffrerential entropy of noise.

$$\begin{aligned} I(X;Y) &= h(Y) - h(Y \mid X) \\ &= h(Y) - h(Z \mid X) \\ &= h(Y) - h(Z) \end{aligned}$$

Capacity of Gaussian Channel

- Without restrictions on input, capacity would be infinity.
- With constraint $E(X^2) \leq P$
  mutual information is maximized when input is Gaussian with variance *P.* Then output is a sum of two independent Gaussians and so it too is Gaussian, with variance *N+P.*
- Differential entropy of Gaussian random variable with varianse σ^2 is $\frac{1}{2}\log_2(2\pi e\sigma^2)$, thus

$$C_G = \frac{1}{2}\log_2(2\pi e(N+P)) - \frac{1}{2}\log_2(2\pi eN)$$

$$= \frac{1}{2}\log_2(1+\frac{P}{N})$$

Capacity of bandlimited Gaussian Channel

- Bandlimited Gaussian channel is a continuous time channel

$$Y = (X + Z) * h_W$$

where where Y is output, *X* is input, *Z* is white noise and h_W is impulse responce of perfect bandpass filter.

- Output can be recconstructed from samples taken at 1/2W interval. Capacity of a sample is achieved when input is Gaussian

$$C_{sample} = \frac{1}{2}\log_2(1 + \frac{P}{WN_0}) \text{ bits/sample}$$

- 2W samples are taken per second, so channel capacity per second is is 2W times sample capacity

$$C = 2WC_{sample} = W \log_2(1 + \frac{P}{2WN_0}) \text{ bits/second}$$

Capacity of RF channel

- In general we can approximate capacity of the RF channel with equation

$$C = \log_2(1 + \rho h^2) \text{ bps/Hz}$$

where *p* is *SNR* and *h* is channel impulse responce.

MIMO channel

- General formula for MIMO channel is

$$Y = H S + n$$

- *H* is *NXM* matrix where *M (N)* is the number of input (output) antennas. Elements h_{ij} of *H* repsesent channel impulse responces between ith input and jth ouptut antenna
- S is Mx1 random variable representing input antennas
- Y is Nx1 random variable representing output antennas

$$H(\tau,t) = \begin{bmatrix} h_{1,1}(\tau,t) & h_{1,2}(\tau,t) & \cdots & h_{1,M}(\tau,t) \\ h_{2,1}(\tau,t) & h_{2,2}(\tau,t) & \cdots & h_{2,M}(\tau,t) \\ \vdots & \vdots & \ddots & \vdots \\ h_{N,1}(\tau,t) & h_{N,2}(\tau,t) & \cdots & h_{N,M}(\tau,t) \end{bmatrix}$$

Capacity of deterministic MIMO channel

- In deterministic MIMO channel matrix is assumed to be non-random
- n is additive temporally white complex zero-mean Gaussian noise with covariance matrix

$$E(nn^H) = N_0 I_M$$

- S is power constrained $\sum_{i=1}^{M} E(s_i^2) \le P$
- Mutual information of output and input is

$$\begin{aligned} I(Y;S) &= h(Y) - h(Y \mid S) \\ &= h(Y) - h(n \mid S) \\ &= h(Y) - h(n) \end{aligned}$$

- Covariance matrix of Y is

$$\begin{aligned} E(YY^H) &= E(HS)E(n)^H + E(HS(HS)^H) + E(n)E(HS)^H + E(nn^H) \\ &= E(HSS^H H^H) + N_0 I_M \\ &= HR_{SS} H^H + N_0 I_M \end{aligned}$$

where *R* is the covariance matrix of *S*

Capacity of deterministic MIMO channel

- Mutual information of input and output is maximized when input is Gaussian => output is Gaussian.
- Differential entropy of multivariate Gaussian is $\frac{1}{2}\log_2((2\pi e)^n |K|)$ where |K| is determinant of the covariance matrix

$$C^D = \log_2(| I_M + \frac{\rho}{M} H R_{SS} H^H |) \text{ bps/Hz}$$

- In the abcense of channel state information at the tranmsitter, it is reasonable to choose *S* to be spatially white i.e. R = I. Then we have

$$C^D = \log_2(| I + \frac{\rho}{M} H H^H |)$$

$$= \sum_{i=1}^{r} \log_2(1 + \frac{\rho}{M}\lambda_i) \text{ bps/Hz}$$

where *r* is the rank of and λ_i denote positive eigenvalues of HH^H.

SISO, MISO, SIMO and MIMO capacities

- For a deterministic channel matrix without channel knowledge at the transmitter we have

$$C_{SISO}^{D} = \log_2(1+\rho h^2) \text{ bps/Hz}$$

$$C_{MISO}^{D} = \log_2(1+\frac{\rho}{M}\sum_{i=1}^{M} h_i^2) \text{ bps/Hz}$$

$$C_{SIMO}^{D} = \log_2(1+\rho\sum_{i=1}^{N} h_i^2) \text{ bps/Hz}$$

$$C_{MIMO}^{D} = \sum_{\mathrm{i}=1}^{\mathrm{r}} \log_2(1+\frac{\rho}{M}\lambda_i) \text{ bps/Hz}$$

- SIMO capacity increases more rapidly with number of receiving antennas than MISO capacity with number of transmitting antennas. This is because MISO has no spatial array gain at the receiver.

MIMO capacity

- At it's best MIMO capacity increase linearly with min(M,N)
- Advantageous of MIMO depends on H; the larger rank and eigenvalues HH have, the more MIMO capacity we have.

Ergodic Capacity of Gaussian MIMO channel with Rayleigh fading

- Additive noise is Gaussian and entries of channel matrix are complex gaussian i.i.d zero mean unit variance random variables.
- Channel matrix is a random variable => mutual information of input and output is a random variable=> "channel capacity" is a random variable **C.**
- Definition: **ergodic capacity** *eC is a* mean of ***C**, eC* = *E(**C**)*.
- **ergodic capacity** is a function of *E(**HH**)*
- By choosing *S* to be i.i.d Gaussian and noting that entries of *HH* are χ^2_{2M} at the diagonal and zero-mean else, we get

$$eC = C^D$$

where C^D is capacity of the deterministic MIMO channel whose channel matrix has $HH^H = E(\chi^2_{2M}) I_N$

Outage Probability

- Channel coding theorem do not apply with ergodic capacity i.e. ergodic capacity do not express maximal zero-error information rate.
- Definition: **Outage probability** for a given rate R is probability that mutual information falls below that rate,

$$P_{out}(R) = P(C \leq R)$$

- Outage probability can be interpreted as PER.

Capacity of frequency-selective fading MIMO channels

- MIMO capacity of frequency flat channel is high
- Frequency band can be divided into subchannels so that bandwith of subchannels is less than coherence bandwith => subchannel is flat
- Assuming that transmit power is allocated uniformly across transmit antennas, mutual information is given by

$$I_{FS} = \frac{1}{K}\sum_{i=1}^{K} \log_2 \left| I_M + \frac{\rho}{N} H_i H_i^H \right| \text{ bps/Hz}$$

and ergodic capacity is $E(I_{FS})$

- Outage probability will in general be lower when frequency band is divided

Summary and conclusions

- Main ideas about deriving channel capacities for discrete, Gaussian, RF and MIMO channels were presented.
- Capacity of RF channel can be approximated by logarithmic function of SNR
- MIMO capacity depends on channel matrix. At it's best MIMO capacity may increase linearly with number of antennas.

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Homeworks

(1) Is it possible to increase channel capacity by channel coding?

(2) Find channel capacities for deterministic multiple antenna channels when SNR = 10dB (=10) and channel matrices are

(a) $H = [1]$ (SISO)

(b) $H = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (SIMO)

(c) $H = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ (MISO)

(d) $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (MIMO) (rank(H) = 3, all three eigenvalues are 1)