Convolutional Coding & Viterbi Algorithm

Er Liu

liuer@cc.hut.fi
Communications Laboratory
16.11.2004

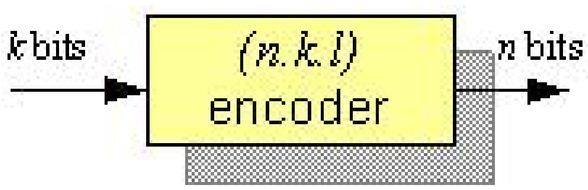
Outline

- **Convolutional Coding**
  - Convolutional code
  - Generator sequence
  - Trellis and state diagram
- **Viterbi Algorithm**
  - Maximum-Likelihood decoding
  - Viterbi algorithm

Convolutional Encoding

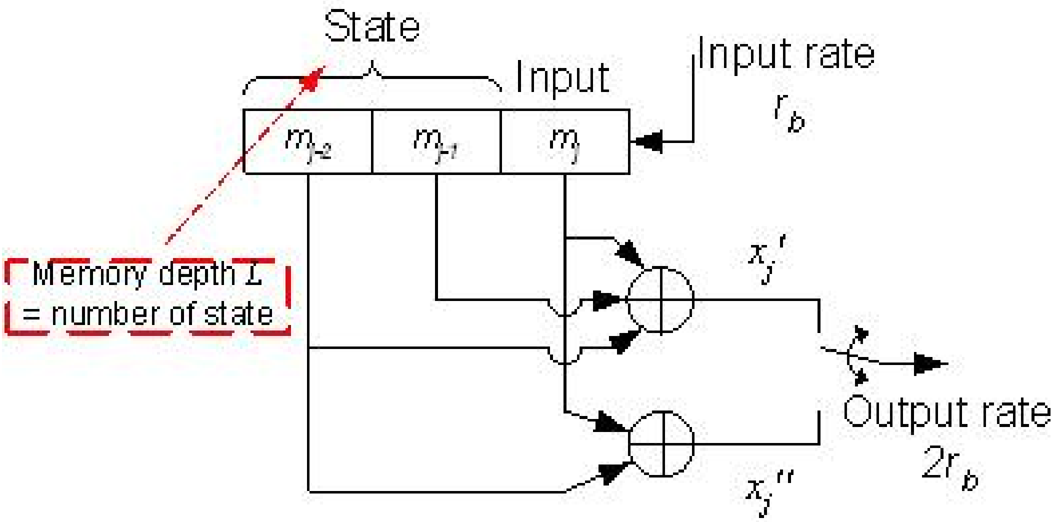
- Convolutional codes are applied in applications that require good performance with low implementation cost. They operate on data stream, not static block.
- Convolutional codes have memory that uses previous bits to encode or decode following bits
- It is denoted by *(n,k,L)*, where *L* is code memory depth
- Code rate *r* is determined by input rate and output rate:

$$r = \frac{r_{input}}{r_{output}} < 1$$



Convolutional Encoder

- Convolutional encoder is a finite state machine (FSM), processing information bits in a serial manner
- Thus the generated code is a function of input and the states of the FSM
- In this *(n,k,L)=(2,1,2)* encoder each message bits influences a span of *n(L+1)=6* successive output bits

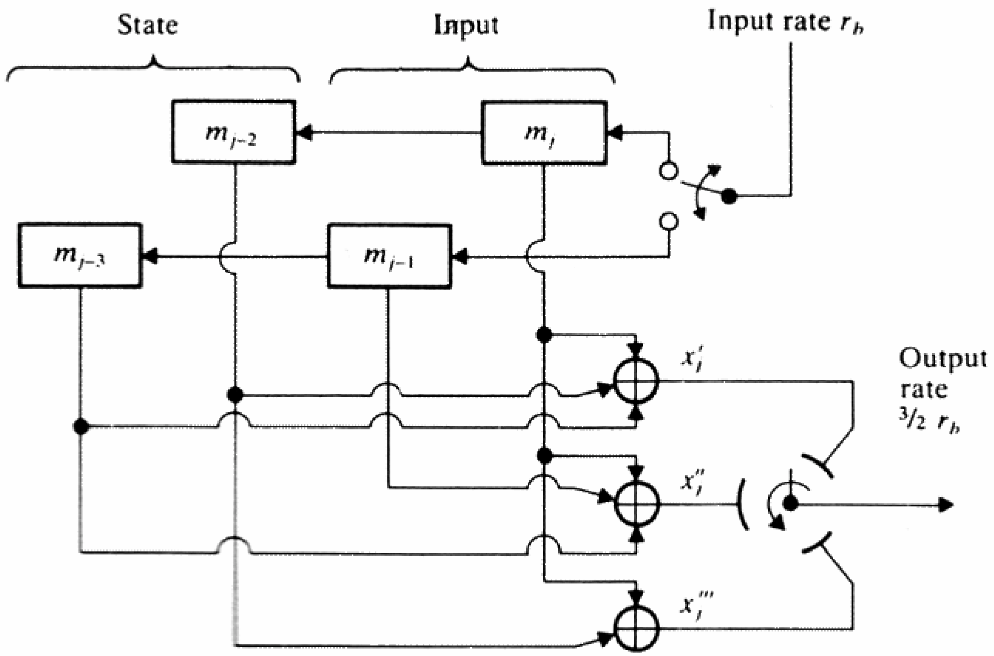


$$x_j' = m_{j-2} \oplus m_{j-1} \oplus m_j$$
$$x_j'' = m_{j-2} \oplus m_j$$

(n,k,L)=(2,1,2) encoder

Another Encoder example

(3,2,1) Convolutional encoder, $k = 2$



$$x'_j = m_{j-3} \oplus m_{j-2} \oplus m_j$$

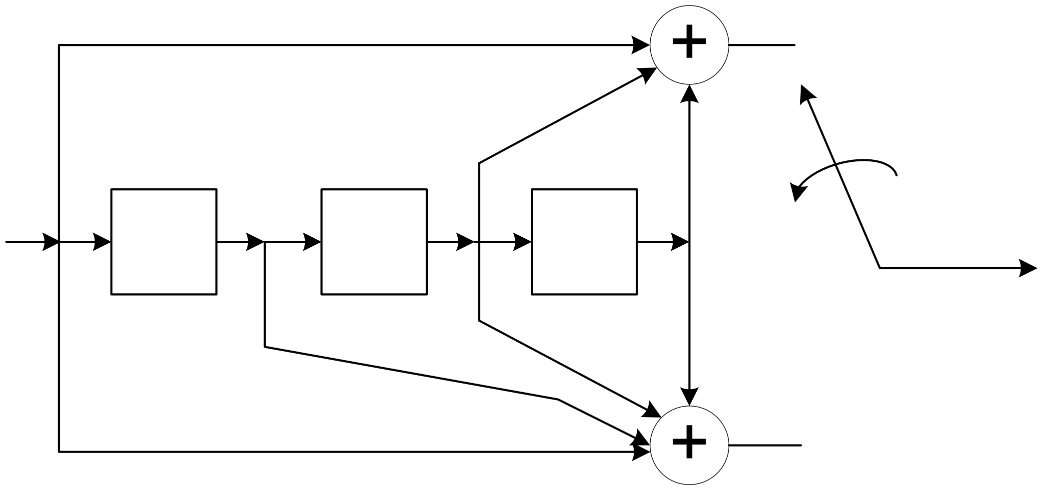
$$x''_j = m_{j-3} \oplus m_{j-1} \oplus m_j$$

$$x'''_j = m_{j-2} \oplus m_j$$

Here each message bit influences a span of $C = n(L+1)=3(1+1)=6$ successive output bits

Generator Sequence

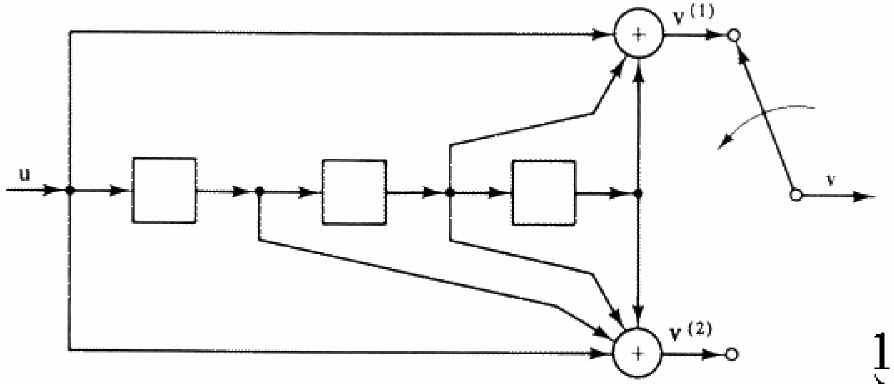
- *(n,k,L)* convolutional code can be described by generator sequences $g^{(1)}$, $g^{(2)}$,..., $g^{(n)}$ that are the impulse responses of each coder output branch



$$\begin{cases} g^{(1)} = [1 \quad 0 \quad 1 \quad 1] \\ g^{(2)} = [1 \quad 1 \quad 1 \quad 1] \end{cases}$$

- Generator sequences specify convolutional code completely by the associated generator matrix
- Encoded convolutional code is produced by matrix multiplication of input and the generator matrix

Example of Using Generator Matrix



$$\begin{pmatrix} \mathbf{g}^{(1)} = [1\ 0\ 1\ 1] \\ \mathbf{g}^{(2)} = [1\ 1\ 1\ 1] \end{pmatrix}$$

$$\underbrace{\underbrace{11 \oplus 00}_{11} \oplus \underbrace{01 \oplus 11}_{10}}_{01} = 01$$

If $\mathbf{u} = (1\ \ 0\ \ 1\ \ 1\ \ 1)$, then

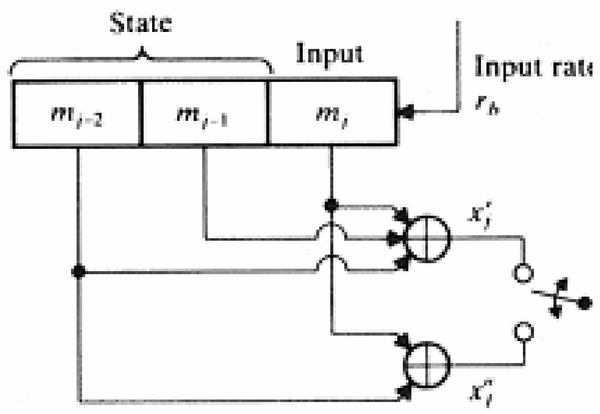
$$\mathbf{v} = \mathbf{u}\mathbf{G}$$

$$= (1\quad 0\quad 1\quad 1\quad 1)\begin{bmatrix} 1\,1 & 0\,1 & 1\,1 & 1\,1 & & & & \\ & 1\,1 & 0\,1 & 1\,1 & 1\,1 & & & \\ & & 1\,1 & 0\,1 & 1\,1 & 1\,1 & & \\ & & & 1\,1 & 0\,1 & 1\,1 & 1\,1 & \\ & & & & 1\,1 & 0\,1 & 1\,1 & 1\,1 \end{bmatrix}$$

$$= (1\ \ 1,\ \ 0\ \ 1,\ \ 0\ \ 0,\ \ \boxed{0\ \ 1,}\ \ 0\ \ 1,\ \ 0\ \ 1,\ \ 0\ \ 0,\ \ 1\ \ 1),$$

It can also use polynomial multiplication

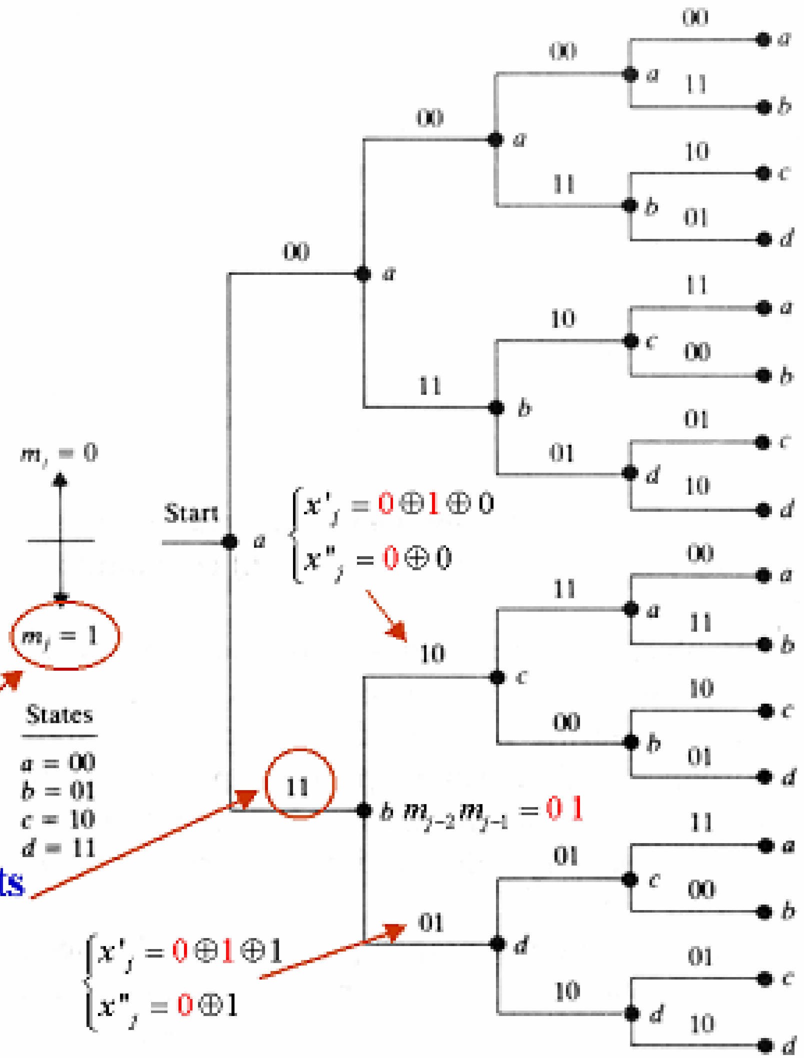
Representation – Code Tree



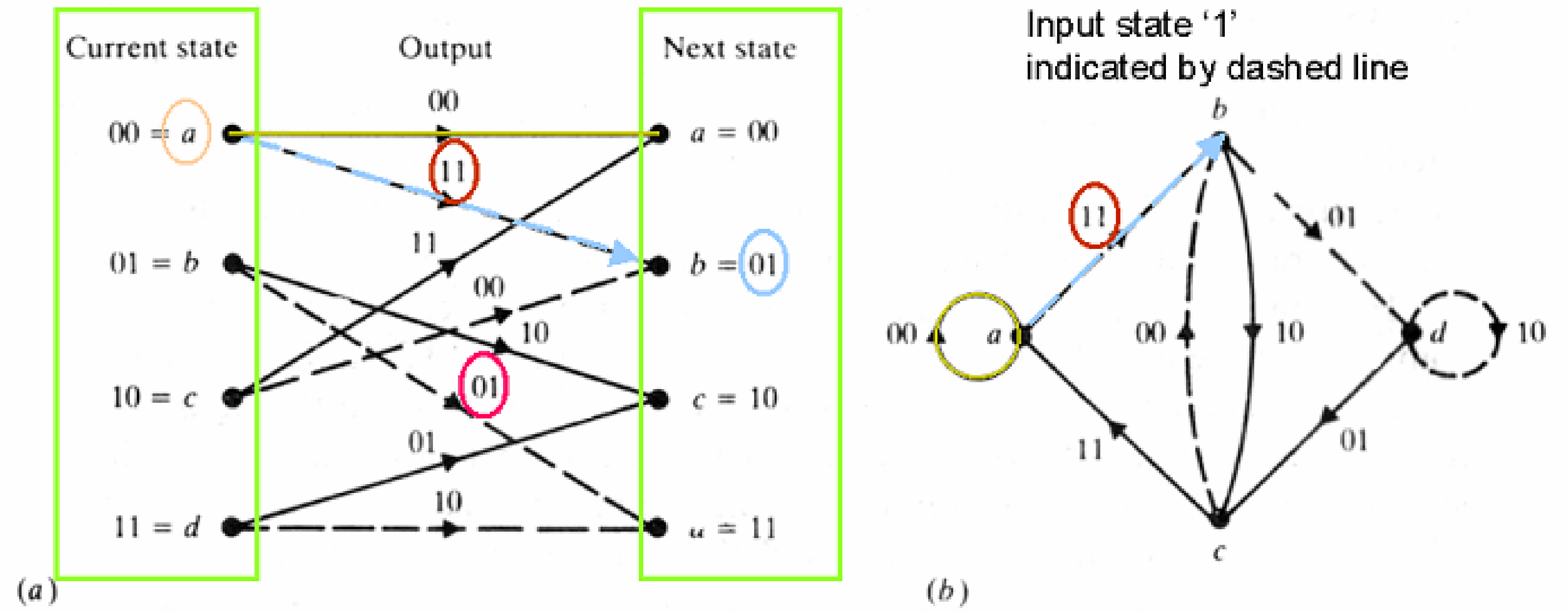
$$\begin{cases} x'_j = m_{j-2} \oplus m_{j-1} \oplus m_j \\ x''_j = m_{j-2} \oplus m_j \end{cases}$$

$$x_{out} = x'_1 x''_1 x'_2 x''_2 x'_3 x''_3 \ldots$$

This tells how **one input bit**
is transformed into **two output bits**
(initially register is all zero)



Trellis and State Diagram



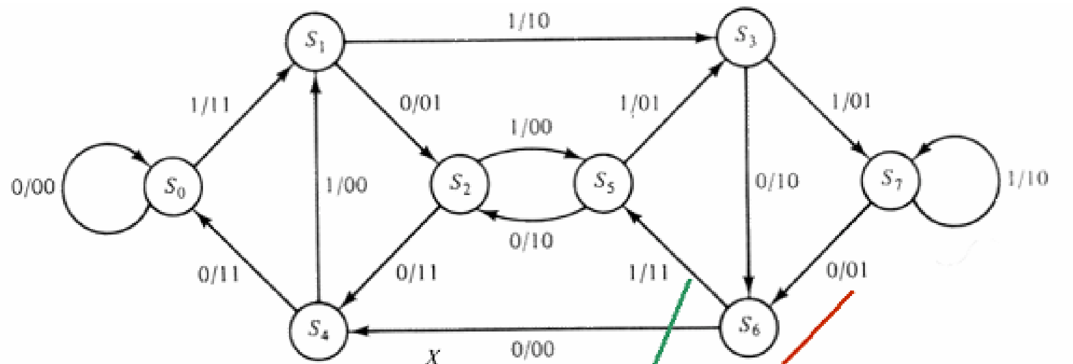
Input	1	1	0	1	1	1	0	0	1	0	0	0	
State	*a*	*b*	*d*	*c*	*b*	*d*	*d*	*c*	*a*	*b*	*c*	*a*	*a*
Output	11	01	01	00	01	10	01	11	11	10	11	00	

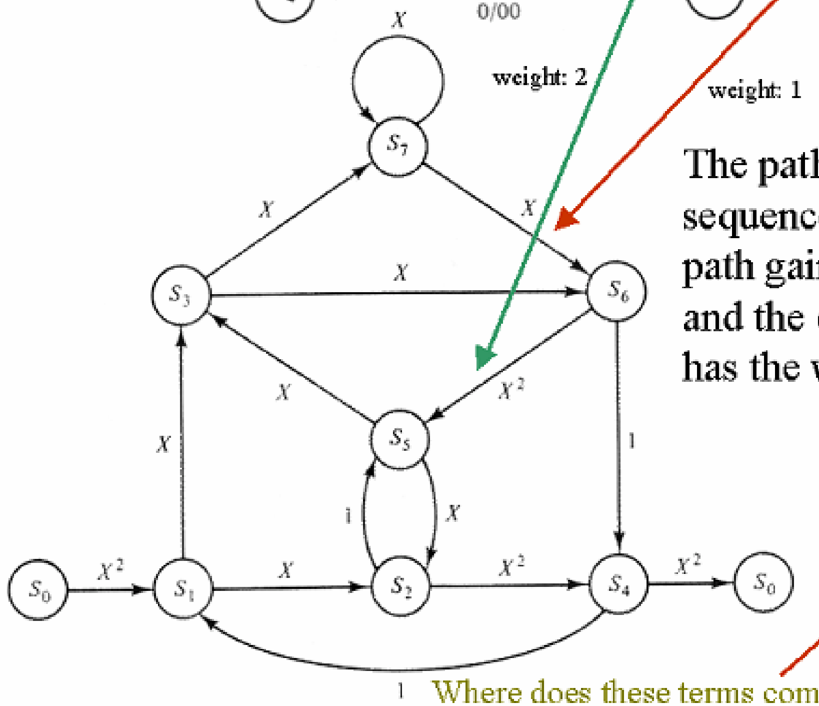
(c)

Shift register states

Er Liu (liuer@cc.hut.fi)

Minimum Hamming Distance





The path representing the state sequence $S_0S_1S_3S_7S_6S_5S_2S_4S_0$ has path gain $X^2X^1X^1X^1X^2X^1X^2X^2=X^{12}$ and the corresponding code word has the weight 12

$$T(X)=\sum_i A_i X^i$$
$$= X^6 + 3X^7 + 5X^8$$
$$+11X^9 + 25X^{10} + \ldots.$$

Where does these terms come from?

Maximum-Likelihood Decoding

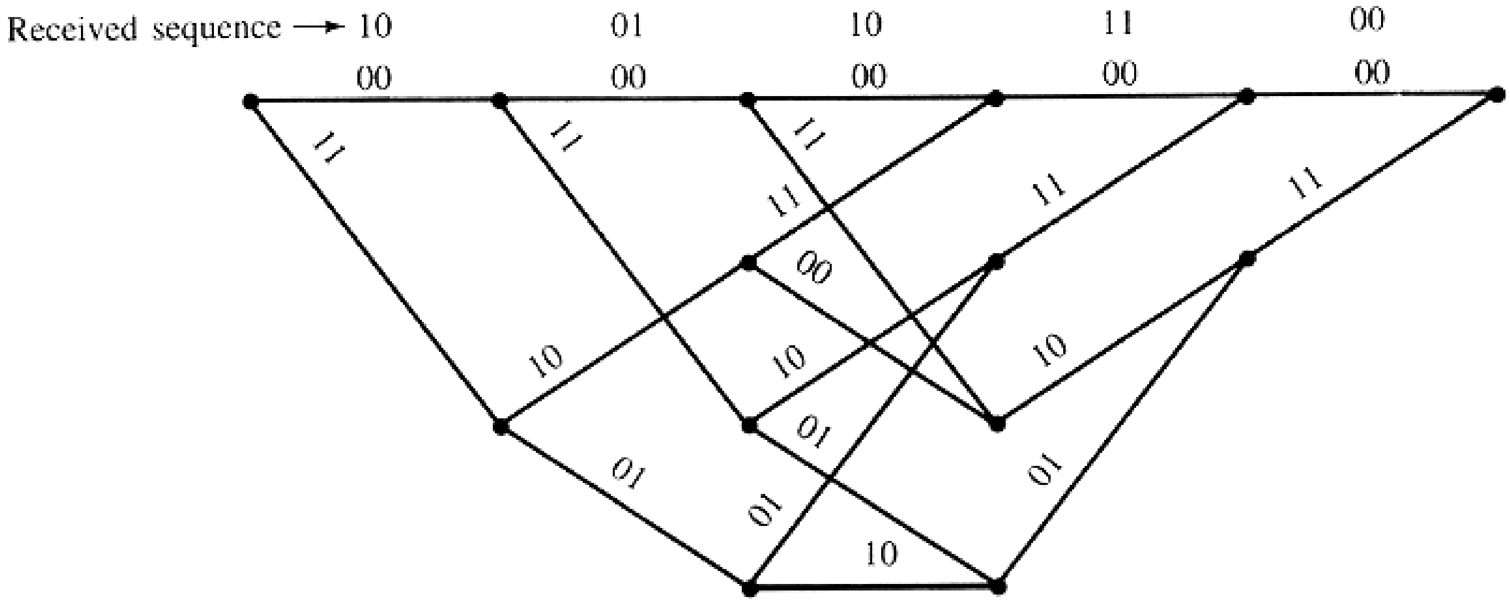
- Maximum likelihood decoding means finding the code branch in the code trellis that was most likely to transmitted
- Therefore maximum likelihood decoding is based on calculating the hamming distances for each branch forming encode word
- Probability to decode sequence is then

$$p(y,x)=\prod_{j=0}^{\infty} p(y_j \mid x_j)$$

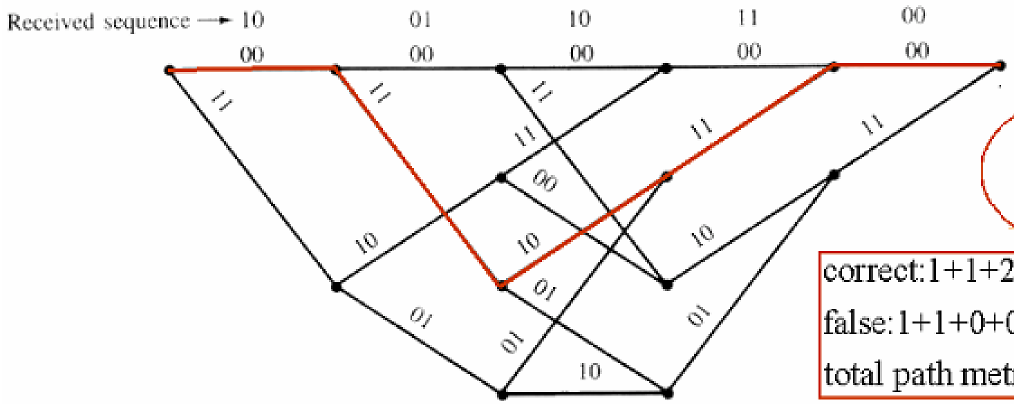
- The most likely path through the trellis will maximize this metric

Example of Maximal Likelihood Detection

- Assume a three bit message is to transmitted. To clear the encoder two zero-bits are appended after message. Thus 5 bits are inserted into encoder and 10 bits produced. Assume channel error probability is p=0.1. After the channel 10,01,10,11,00 is produced. What comes after decoder, e.g. what was most likely the transmitted sequence?



Example of Maximal Likelihood Detection



$5(-2.3)+5(-0.11)=-12.05$

errors correct

correct: $1+1+2+2+2=8; 8\cdot(-0.11)=-0.88$

false: $1+1+0+0+0=2; 2\cdot(-2.30)=-4.6$

total path metric: -5.48

The Hamming distance between this path and the received sequence is 5. All paths (specified by the encoder input bits) and their path metrics and Hamming distances are listed below.

The largest metric, verify that you get the same result!

Received sequence: 10, 01, 10, 11, 00

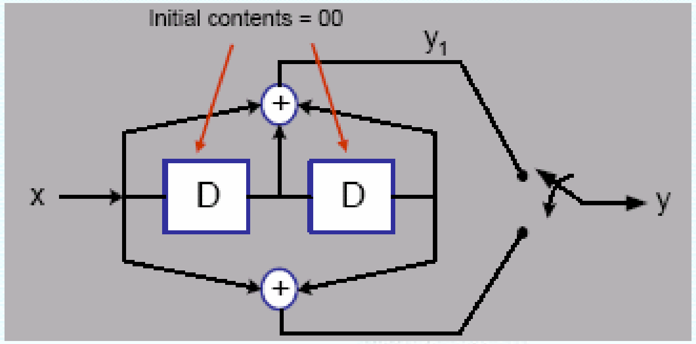
Note also the Hamming distances!

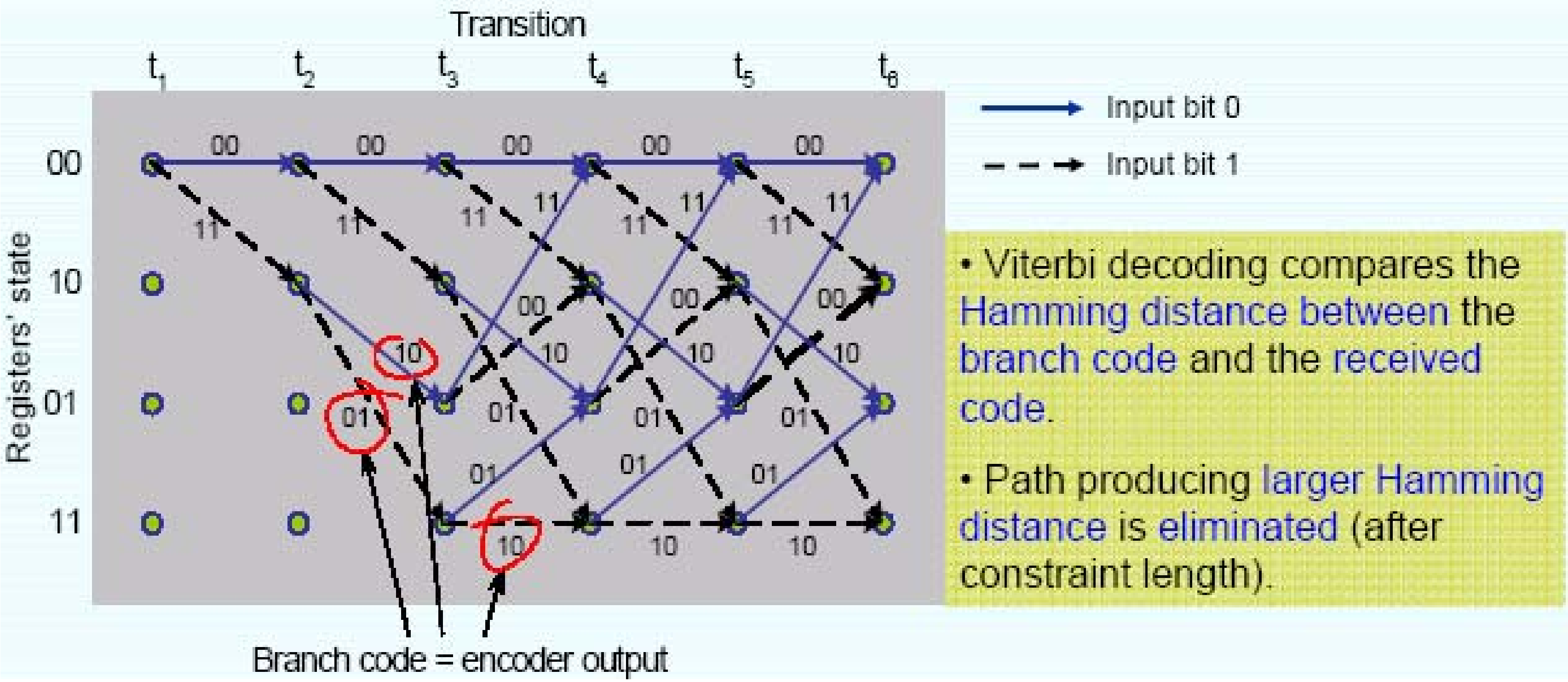
Path	Code Sequence	Path Metric	Hamming Distance
0, 0, 0, 0, 0	00, 00, 00, 00, 00	−12.05	5
0, 0, 1, 0, 0	00, 00, 11, 10, 11	−14.24	6
0, 1, 0, 0, 0	00, 11, 10, 11, 00	−5.48	2
0, 1, 1, 0, 0	00, 11, 01, 01, 11	−16.43	7
1, 0, 0, 0, 0	11, 10, 11, 00, 00	−14.24	6
1, 0, 1, 0, 0	11, 10, 00, 10, 11	−16.43	7
1, 1, 0, 0, 0	11, 01, 01, 11, 00	−7.67	3
1, 1, 1, 0, 0	11, 01, 10, 01, 11	−9.86	4

Viterbi Algorithm

- ML algorithm is too complex to search all available pathes
  - End to end calculation
- Viterbi algorithm performs ML decoding by reducing its complexity
  - Eliminate least likely trellis path at each transmission stage
  - Reduce decoding complexity with early rejection of unlike pathes
- Viterbi algorithm gets its efficiency via concentrating on *suvival paths* of the trellis

Viterbi decoding

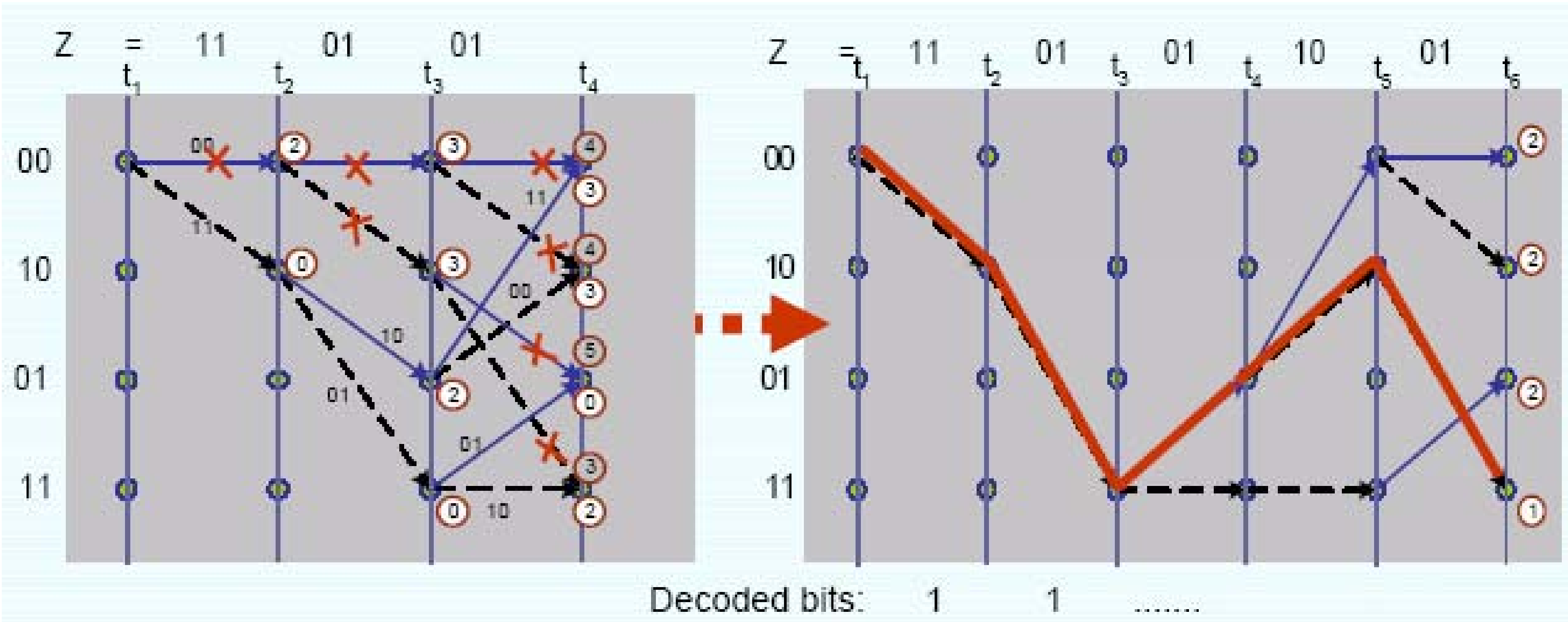




- Viterbi decoding compares the Hamming distance between the branch code and the received code.
- Path producing larger Hamming distance is eliminated (after constraint length).

Example of Viterbi decoding

Input data : m =1 1 0 1 1
Codeword : X = 11 01 01 00 01
Received code : Z = 11 01 01 10 01

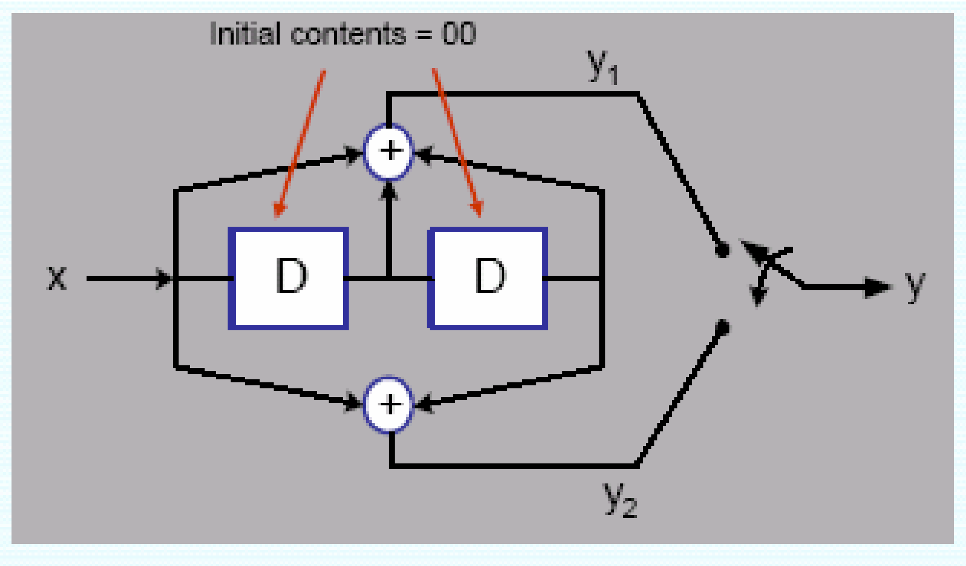


Er Liu (liuer@cc.hut.fi)

Homework

- Please use Viterbi algorithm to decode the received sequence:

  Z=[11 10 10 10 01]



Please draw the trellis and state diagram

Any questions?

Thanks!