S.72.333 Post-graduate Seminar in Radio Communications

Single Symbol Optimum Receiver Principles Part I

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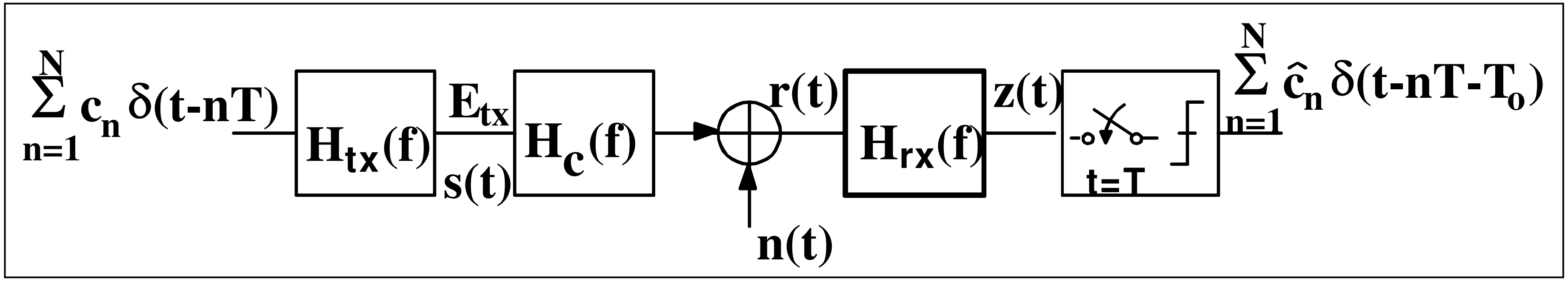
THE BASIC DIGITAL TRANSMISSION SYSTEM



- $\{c_n\}$ is the sequence of transmitted data symbols
- s(t) is the transmitted signal, where $h_{tx}(t)$ determines the transmitted pulse for each symbol, the average transmitted symbol energy is E_{tx}
- $H_c(f)$ is the transfer function of the transmission channel which distorts the transmitted pulse
- r(t) is the received signal distorted by the transmission channel and perturbed by additive noise
- $H_{rx}(f)$ is the receiver filter
- z(t) is the input signal to the sampler and decision circuit
- $\{\hat{c}_n\}$ is the sequence of detected data symbols in the receiver output
- T_o is the total transmission delay
- This is a very simplified and idealized transmission system model

Problem definition

1° The noise causes uncertainty in the receiver about which symbol is actually transmitted

2° The channel distortion causes the used pulse to overlap to other symbol intervals → intersymbol interference in symbol sequence transmission → additional uncertainty about the transmitted symbol

3° A concequence of noise and inter-symbol interference is that the receiver symbol decision circuit will sometimes produce symbol errors

4° Due to the random nature of noise and the transmitted symbols causing ISI, also the symbol errors occurs in a random manner → system performance must be characterized by probabilistic measures and system analysis is done with statistical methods

5° The design of an optimum receiver is aimed to minimize the probability of symbol errors (bit errors)

6° This is achieved by finding the optimum symbol decision algorithm and the optimum receiver filter (and transmitter filter)

7° In practice non-ideal implementation or the use of sub-optimal solutions will cause performance degradation and the optimum receiver gives a reference performance to which actual performance can be compared

System models

❒Transmission models

- single symbol
- symbol sequence: burst, continuous, full response, partial response
- binary/M-ary transmission
- parallel transmission: MIMO, MISO, SIMO, SISO
- uncoded/channel coded transmission

❒Channel models

- AWGN-channel (Additive White Gaussian Noise) TI/FF, TV/FF, TI/SF, TV/SF
- ACGN-channel (Additive Colored Gaussian Noise) TI/FF, TV/FF, TI/SF, TV/SF

❒Receiver models

- symbol-by-symbol detection
- symbol sequence detection
- Equalizer receiver
- Diversity receiver
- RAKE-receiver
- MUD-receiver

DIGITAL REFERENCE RECEIVERS

(Those marked by "■" are treated more in detail)

- � **Single symbol optimum receiver in the AWGN-channel**
- ❒ **Single symbol optimum receiver in the ACGN-channel**
- � **Symbol sequence receiver in the AWGN-channel with symbol-by-symbol decision**
- ■ **Symbol sequence receiver in the band-limited AWGN-channel with symbol-by-symbol decision, ISI-elimination with raised-cosine filtering**
- ❒ **Symbol sequence receiver in the band-limited AWGN-channel with symbol-by-symbol decision, controlled ISI elimination with partial response signalling**
- ❒ **Symbol sequence receiver in the band-limited AWGN-channel with symbol-by-symbol decision, MSE-approach**
- ■ **Symbol sequence optimum receiver in the band-limited AWGN-channel**

SINGLE SYMBOL OPTIMUM RECEIVER IN THE AWGN-CHANNEL: Outline of the treatment

- **Derivation of the MAP decision rule given a known received *signal vector* in a channel with additive noise**

  $\{r_k\} \to \hat{c}_i : MAX\left\{P(c_i)\, p\left(\{r_k\} \middle| c_i\right)\right\} \to$ **decision areas D_i**

- **Derivation of the symbol error probability**

  $P(e|c_i) = P\left(\{r_k\} \in D_j \middle| c_i, i \neq j\right) \quad P(e) = \sum_i P(c_i) P(e|c_i)$

- **Conversion of received pulse waveforms into signal vectors using orthonormal function expansion**

  $r_k = \int r(t)\phi_k(t)dt \quad \int \phi_i(t)\phi_j(t)dt = \delta_{ij}$

- **The concept of sufficient statistics and the choice of orthogonal functions**
- **The optimum receiver in the AWGN-channel**

  correlation receiver ⇔ matched filter receiver

- **PAM-signalling and its performance in the AWGN-channel**
- **Orthogonal signalling and its performance in the AWGN-channel**

THE OPTIMUM DECISION RULE IN A VECTOR CHANNEL CONTAINING ADDITIVE NOISE

The output of a vector channel containing additive noise is a N-component signal sample vector $R = S_i + N$, where

$$R = \{r_1, r_2, ..., r_N\}^T \ , \quad S_i = \{s_{i1}, s_{i2}, ..., s_{iN}\}^T \ , \quad N = \{n_1, n_2, ..., n_N\}^T \qquad \textbf{(1a,b,c)}$$

The optimum decision rule maximizes the average probability of a correct symbol decision and minimizes the average symbol error probability. The average probability of a correct symbol decision is

$$P(c) = \int\int ... \int p(R) P\left(c \middle| R\right) dR \qquad \textbf{(2)}$$

Because the joint density function and the conditional probability of a correct decision are positive, the average probability is maximized if the conditional probability is maximized for each possible R-vector.

The conditional probability of a correct decision can be expressed as

$$P\left(c \middle| R\right) = P\left(\hat{c} = c_i \middle| R, c_i\right) \tag{3}$$

The optimum decision rule calculates the a posteriori probabilities $P\left(\hat{c} = c_k \middle| R\right)$ for all symbols c_k and chooses the symbol that maximizes the a posteriori probability. This gives the Maximum A Posteriori (MAP) decision rule

$$\hat{c} = c_k \Rightarrow \underset{k}{MAX}\left\{P\left(\hat{c} = c_k \middle| R\right)\right\} \tag{4}$$

The MAP-decision rule in this form is not very practical because of the difficulty to calculate the a posteriori probabilities. Application of Bayes' rule gives a more tractable expression:

$$P\left(\hat{c} = c_k \middle| R\right) = P\left(c_k \middle| R\right) = \frac{P\left(c_k\right) p\left(R \middle| c_k\right)}{p(R)} \tag{5}$$

There is no need to use the joint density function of R when comparing the different c_k-values, so the decision rule takes its final form

$$\boxed{\hat{c} = c_k \Rightarrow \underset{k}{MAX}\left\{P(c_k)\, p\left(R \middle| c_k\right)\right\}, k = 1,2,\ldots,M} \quad (6)$$

- $P(c_k)$ is the a priori probability of the possible symbol values. In most cases $P(c_k) = 1/M$.
- $p\left(R \middle| c_k\right)$ is the joint density function of the received vector samples conditioned on the symbol value

There is another, sub-optimum decision rule which doesn't consider the symbol probabilities. That is called the Maximum Likelihood (ML) decision rule

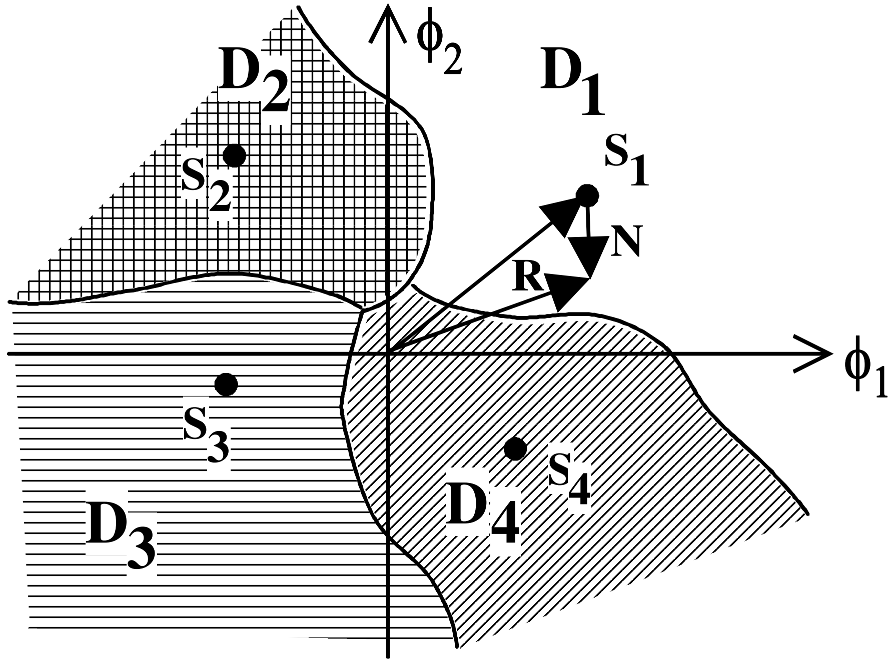
$$\hat{c} = c_k \Rightarrow \underset{k}{MAX}\left\{p\left(R \middle| c_k\right)\right\}, k = 1,2,\ldots,M \quad (7)$$

The ML-receiver is more easily implemented than the MAP-receiver. For equally probable symbol values their performance is also equal. In practice the degradation is minor unless very large unbalance in the symbol probabilities. From the MAP decision rule follows that the vector signal space can be divided into separate decision areas, in each of which a certain symbol

maximizes the posterior probability. The borders between the decision areas D_i and D_j can be solved from the equality

$$P(c_i)\,p(R|c_i) = P(c_j)\,p(R|c_j) \quad (8)$$

In the decision area D_1 the symbol c_1 maximizes the posterior probability. If the noise vector brings the signal vector R e.g. to the decision area D_2 the receiver will make an erraneous decision. The error probability is a measure of the receiver performance.



GENERAL EXPRESSION OF THE SYMBOL ERROR PROBABILITY

If the receiver always makes a decision (also other reception strategies may be used) then on a given transmitted symbol the error event is defined in the following way:

$$\{e|c_i\} = \{R \in D_j |c_i\}, \quad j \neq i \tag{9}$$

Correspondingly the event of a correct decision is defined:

$$\{c|c_i\} = \{R \in D_i |c_i\} \tag{10}$$

It is obvious that the union of the both events form the entire decision space:

$$\{c|c_i\} + \{e|c_i\} = \{R \in D_i |c_i\} + \{R \notin D_i |c_i\} = \{S|c_i\} \tag{11}$$

and consequently

$$P\{e|c_i\} = 1 - P\{c|c_i\} \tag{12}$$

When the channel contains additive noise the SEP can be calculated in two ways:

$$P\left\{e \middle| c_i\right\} = \underset{S-D_i}{\int\int\ldots\int} p_n(R - S_i)dR \tag{13}$$

or

$$P\left\{e \middle| c_i\right\} = 1 - \underset{D_i}{\int\int\ldots\int} p_n(R - S_i)dR \tag{14}$$

Because the decision areas are separate the average SEP can be calculated with the total probability formula:

$$P(e) = \sum_{k=1}^{M} P\left(c_k\right)P\left(e \middle| c_k\right) \tag{15}$$

CONVERSION OF RECEIVED PULSE WAVEFORMS INTO SIGNAL VECTORS USING ORTHONORMAL FUNCTION EXPANSION

In practice digital transmission takes place using analog pulse waveforms, so that a different pulse is used for each symbol value. The MAP-receiver however, bases its decision on signal vectors. Now the receiver must be able to convert the received time-continuous pulse into a signal vector preferably with as few components as possible.
One way to do this is to use suitable functional representations of the signals involved. In this case the pulses can be expanded into series:

$$s_i(t) = \sum_{k=1}^{K} s_{ik}\phi_k(t) \qquad \textbf{(16)}$$

where $\phi_k(t)$ are the base functions and the vector components can be calculated by correlating the pulse signal with the base functions:

$$s_{ik} = \int s_i(t)\phi_k(t)dt \qquad \textbf{(17)}$$

Next problem is how to choose the base functions. It appears that SEP-calculations are more convenient if the base functions are chosen so that the noise components are statistically independent and their statistical properties are easily determined.
In the AWGN-channel this is achieved if the base functions form an orthonormal function set. An orthonormal set fulfils the condition

$$\int_0^T \phi_i(t)\phi_j(t)dt = \delta_{ij} \qquad (18)$$

In the more general ACGN-channel the base functions are solutions to the integral equation

$$\int_0^T R_n(t-u)\phi_k(u)du = \lambda_k \phi_k(t) \qquad (19)$$

where $R_n(\tau)$ is the autocorrelation function of the additive noise and λ_k is the eigenvalue corresponding to the eigenfunction $\phi_k(t)$.

Next the desired properties of the orthonormal function expansion in the AWGN-channel are demonstrated.

The noise vector components are

$$n_k = \int n(t)\phi_k(t)dt \qquad (20)$$

It follows that the expectation of the noise components are

$$E\{n_k\} = \int E\{n(t)\}\phi_k(t)dt \qquad (21)$$

so zero-mean noise produces zero-mean noise components.

The cross-covariance of two noise components is

$$E\{n_k n_l\} = E\left\{\int n(t)\phi_k(t)dt \int n(u)\phi_k(u)du\right\} = \int\int E\{n(t)n(u)\}\phi_k(t)\phi_l(u)dtdu$$
$$= \int\int \frac{N_o}{2}\delta(t-u)\phi_k(t)\phi_l(u)dtdu = \frac{N_o}{2}\int \phi_k(t)\phi_l(t)dt = \frac{N_o}{2}\delta_{kl} \qquad (22)$$

It appears that the noise components are uncorrelated and therefore statistically independent at least in the Gaussian case and of equal power.

THE CONCEPT OF SUFFICIENT STATISTICS AND THE CHOICE OF ORTHOGONAL FUNCTIONS

Because of the AWGN an infinite number of orthonormal function would be needed to completely represent the received signal. If, however, only a finite number of functions is needed to represent all the used pulse wave forms, or

$$s_{ik} \equiv 0, \quad k > K \tag{23}$$

then due to the statistical independence of the noise vector components the density function of the received signal vector conditioned on a given transmitted symbol is:

$$p\left(R \middle| c_i\right) = p_n\left(r_1 - s_{i1}\right) p_n\left(r_1 - s_{i1}\right) \cdot \ldots \cdot p_n\left(r_K - s_{iK}\right) p_n\left(r_{K+1}\right) p_n\left(r_{K+2}\right) \cdot \ldots \tag{24}$$

But in the comparison of the posterior probability no attention needs to be paid to the factors not depending on *i*, so the *K* first components form a sufficient statistics for decision.
A minimum set of orthonormal functions can be obtained with the Gram-Schmidt algorithm. Often clever reasoning will do as well.

The Gram-Schmidt algorithm

The signal set to be represented is $\{s_i(t)\}, i = 1,2,...,M$, where all signals vanish outside the time interval [-T/2, T/2]
Definitions:

Signal norm: $\|x(t)\| = \sqrt{\int_T x^2(t)dt} = \sqrt{E_x}$ **(25)**

Signal correlation: $\rho_{x,y} = \int_T x(t)y(t)dt$ **(26)**

The first orthonormal function is obtained by normalizing an arbitrary signal function. e.g.

$$\phi_1(t) = \frac{s_1(t)}{\|s_1(t)\|} \quad (27)$$

The second orthonormal function is obtained by taking next arbitrary signal function, subtracting its projection on the previous orthonormal function, and normalizing the difference, e.g.

$$\phi_2(t) = \frac{s_2(t) - \rho_{s_2,\phi_1} \cdot \phi_1(t)}{\left\| s_2(t) - \rho_{s_2,\phi_1} \cdot \phi_1(t) \right\|} \qquad \textbf{(26)}$$

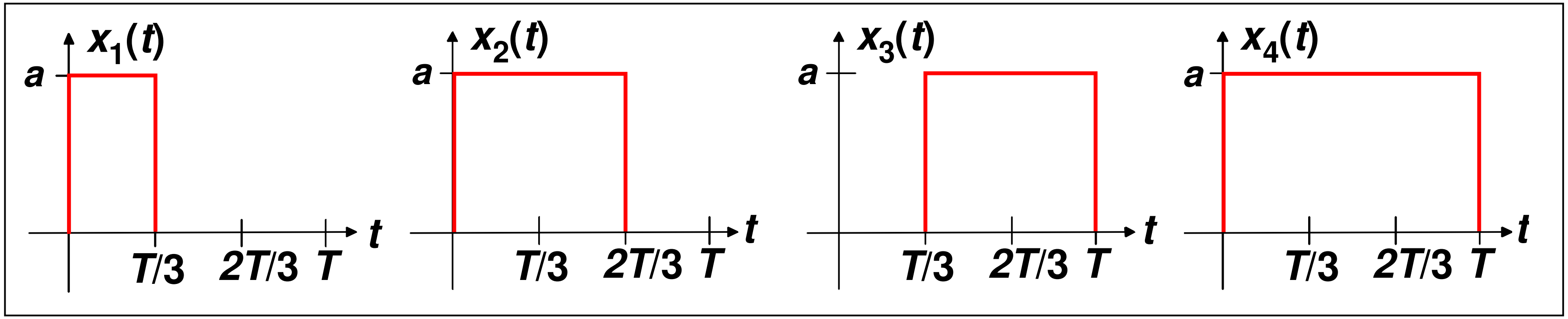
The rest of the orthonormal function are obtained by taking the next arbitrary signal function, subtracting from that its projection on all previous orthonormal functions, and dividing the difference with its norm, i.e.

$$\phi_k(t) = \frac{s_k(t) - \sum_{i=1}^{k-1} \rho_{s_k,\phi_i} \cdot \phi_i(t)}{\left\| s_k(t) - \sum_{i=1}^{k-1} \rho_{s_k,\phi_i} \cdot \phi_i(t) \right\|} \qquad \textbf{(27)}$$

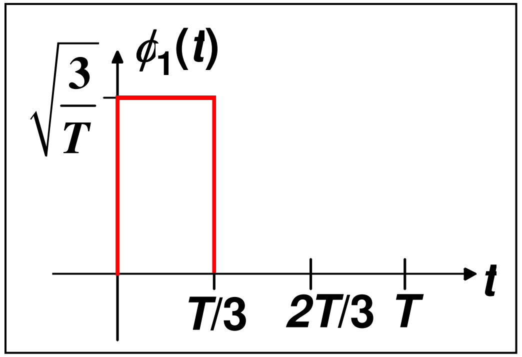
If the minimum set is already obtained, Eq. (27) will produce zeroes.

Example 1. Application of the Gram-Schmidt algorithm

Original signal set, M = 4



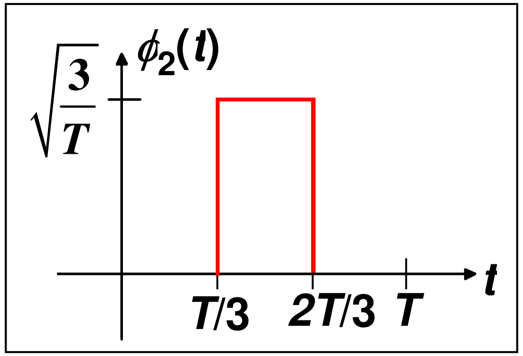
$$\phi_1(t) = \frac{x_1(t)}{\|x_1(t)\|} = \frac{a\,\mathrm{rect}\left(\frac{t - T/6}{T/3}\right)}{\sqrt{\int_0^{T/3} a^2 dt}} = \sqrt{\frac{3}{T}}\,\mathrm{rect}\left(\frac{t - T/6}{T/3}\right)$$



$$\rho_{x_2,\phi_1} = \int_0^{T/3} a\sqrt{\frac{3}{T}}dt = a\sqrt{\frac{T}{3}}$$

$$\phi_2(t) = \frac{x_2(t) - \rho_{x_2,\phi_1} \cdot \phi_1(t)}{\left\| x_2(t) - \rho_{x_2,\phi_1} \cdot \phi_1(t) \right\|} = \frac{x_2(t) - a\sqrt{\frac{T}{3}} \cdot \phi_1(t)}{\left\| x_2(t) - a\sqrt{\frac{T}{3}} \cdot \phi_1(t) \right\|}$$

$$= \frac{a \,\mathrm{rect}\left(\frac{t - T/2}{T/3}\right)}{\sqrt{\int_{T/3}^{2T/3} a^2 dt}} = \sqrt{\frac{3}{T}}\,\mathrm{rect}\left(\frac{t - T/2}{T/3}\right)$$



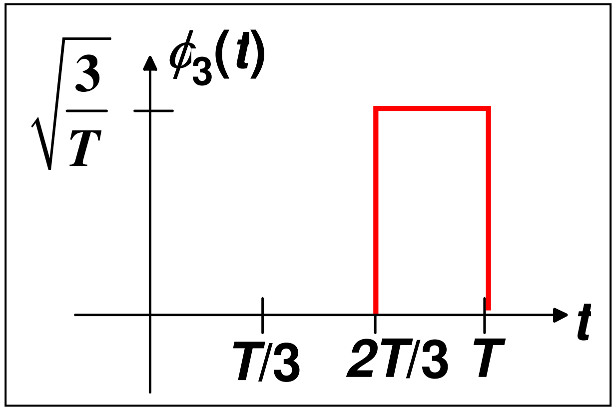
The next last expression is obtained by graphical inspection

As the duration and amplitudes of both orthonormal functions are equal, and the amplitude of the third signal is constant the correlation are equally large:

$$\rho_{x_3,\phi_1} = \rho_{x_3,\phi_2} = a\sqrt{\frac{T}{3}}$$

$$\phi_3(t)=\frac{x_3(t)-\rho_{x_3,\phi_1}\cdot\phi_1(t)-\rho_{x_3,\phi_2}\cdot\phi_2(t)}{\left\|x_3(t)-\rho_{x_3,\phi_1}\cdot\phi_1(t)-\rho_{x_3,\phi_2}\cdot\phi_2(t)\right\|}=\frac{x_3(t)-a\sqrt{\frac{T}{3}}\cdot\left(\phi_1(t)+\phi_2(t)\right)}{\left\|x_3(t)-a\sqrt{\frac{T}{3}}\cdot\left(\phi_1(t)+\phi_2(t)\right)\right\|}$$

$$=\frac{a\,\mathrm{rect}\left(\frac{t-5T/6}{T/3}\right)}{\sqrt{\int_{2T/3}^{T}a^2dt}}=\sqrt{\frac{3}{T}}\,\mathrm{rect}\left(\frac{t-5T/6}{T/3}\right)$$



Again the next last expression is obtained by graphical inspection

As above the correlations

$$\rho_{x_4,\phi_1}=\rho_{x_4,\phi_2}=\rho_{x_4,\phi_3}=a\sqrt{\frac{T}{3}}$$

$$\phi_4(t) = \frac{x_4(t) - \rho_{x_4,\phi_1} \cdot \phi_1(t) - \rho_{x_4,\phi_2} \cdot \phi_2(t) - \rho_{x_4,\phi_3} \cdot \phi_3(t)}{\left\| x_4(t) - \rho_{x_4,\phi_1} \cdot \phi_1(t) - \rho_{x_4,\phi_2} \cdot \phi_2(t) - \rho_{x_4,\phi_3} \cdot \phi_3(t) \right\|}$$

$$= \frac{x_4(t) - a\sqrt{\frac{T}{3}} \cdot \left(\phi_1(t) + \phi_2(t) + \phi_3(t)\right)}{\left\| x_4(t) - a\sqrt{\frac{T}{3}} \cdot \left(\phi_1(t) + \phi_2(t) + \phi_3(t)\right) \right\|} = 0$$

The four signals can thus be represented with only three orthonormal functions, which is the minimum set.

The same size will be obtained regardless of in which order the signals are selected, but the orthonormal function set will be different.

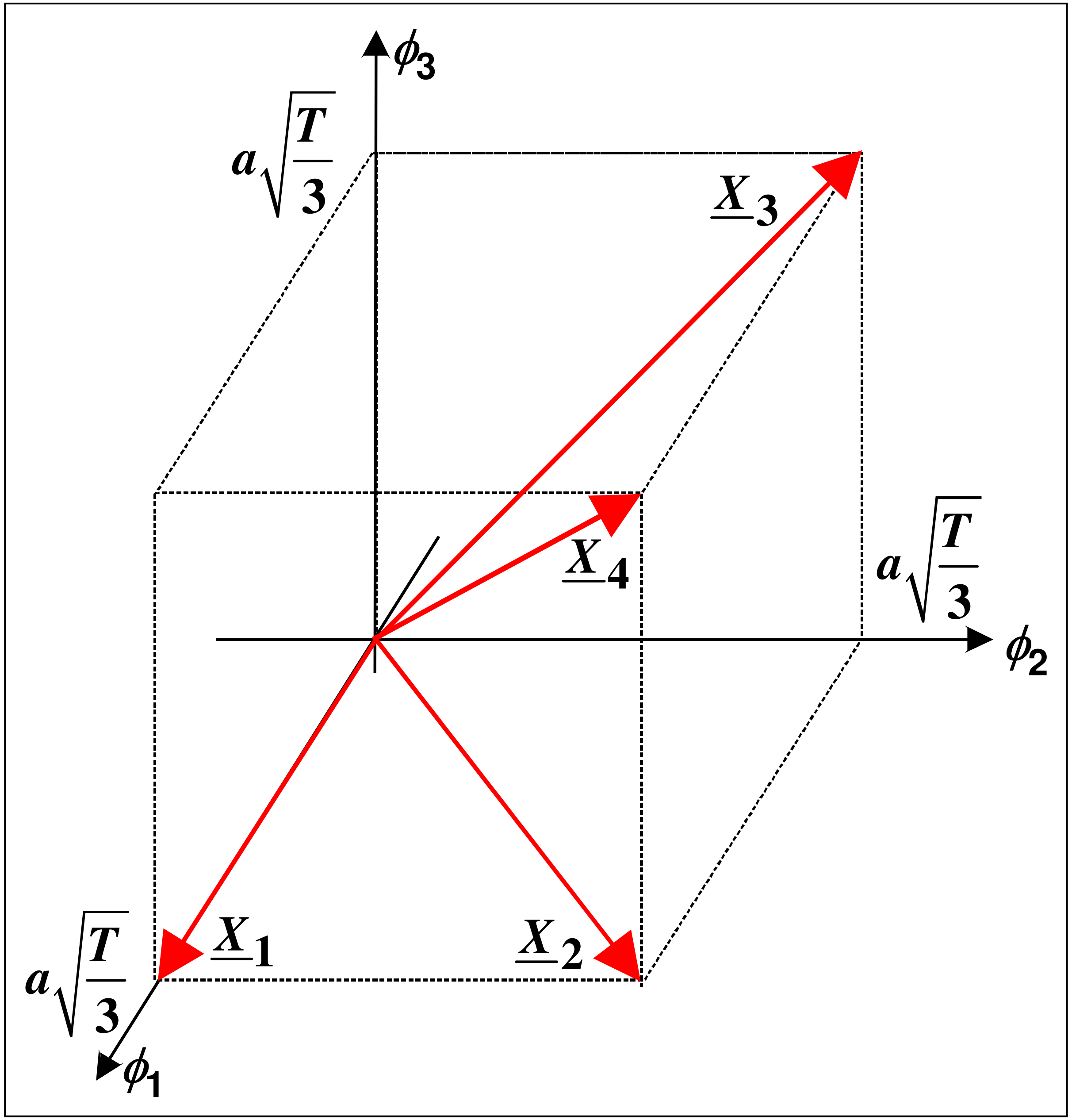
The signal vectors are:

$$\underline{X}_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} \int x_1(t)\phi_1(t)dt \\ \int x_1(t)\phi_2(t)dt \\ \int x_1(t)\phi_3(t)dt \end{bmatrix} = \begin{bmatrix} \int_0^{T/3} a\sqrt{3/T}\,dt \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a\sqrt{T/3} \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{X}_2 = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} \int x_2(t)\phi_1(t)dt \\ \int x_2(t)\phi_2(t)dt \\ \int x_2(t)\phi_3(t)dt \end{bmatrix} = \begin{bmatrix} \int_0^{T/3} a\sqrt{3/T}\,dt \\ \int_{T/3}^{2T/3} a\sqrt{3/T}\,dt \\ 0 \end{bmatrix} = \begin{bmatrix} a\sqrt{T/3} \\ a\sqrt{T/3} \\ 0 \end{bmatrix}$$

$$\underline{X}_3 = \begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix} = \begin{bmatrix} \int x_3(t)\phi_1(t)dt \\ \int x_3(t)\phi_2(t)dt \\ \int x_3(t)\phi_3(t)dt \end{bmatrix} = \begin{bmatrix} 0 \\ \int_{T/3}^{2T/3} a\sqrt{3/T}\,dt \\ \int_{2T/3}^{T} a\sqrt{3/T}\,dt \end{bmatrix} = \begin{bmatrix} 0 \\ a\sqrt{T/3} \\ a\sqrt{T/3} \end{bmatrix}$$

$$\underline{X}_4 = \begin{bmatrix} x_{41} \\ x_{42} \\ x_{43} \end{bmatrix} = \begin{bmatrix} \int x_4(t)\phi_1(t)dt \\ \int x_4(t)\phi_2(t)dt \\ \int x_4(t)\phi_3(t)dt \end{bmatrix} = \begin{bmatrix} \int_{0}^{T/3} a\sqrt{3/T}\,dt \\ \int_{T/3}^{2T/3} a\sqrt{3/T}\,dt \\ \int_{2T/3}^{T} a\sqrt{3/T}\,dt \end{bmatrix} = \begin{bmatrix} a\sqrt{T/3} \\ a\sqrt{T/3} \\ a\sqrt{T/3} \end{bmatrix}$$



ϕ_3
$a\sqrt{\frac{T}{3}}$
$\underline{X}_3$
$\underline{X}_4$
$a\sqrt{\frac{T}{3}}$
ϕ_2
$a\sqrt{\frac{T}{3}}$
$\underline{X}_1$
$\underline{X}_2$
ϕ_1

THE OPTIMUM RECEIVER IN THE AWGN-CHANNEL

It is assumed that K vector components form a sufficient statistics. With AWGN the conditional density function of the received signal vector is

$$p\left(R\middle|c_i\right)=\prod_{k=1}^{K}\frac{1}{\sqrt{2\pi N_o/2}}e^{-\frac{\left(r_k-s_{ik}\right)^2}{2N_o/2}}=\frac{1}{\left(\pi N_o\right)^{K/2}}e^{-\frac{1}{N_o}\sum_{k=1}^{K}\left(r_k-s_{ik}\right)^2} \qquad (28)$$

Because lnf(x) is a monotonic function of f(x), it is maximized for the same x-value as f(x). Logarithmization of the decision function brings some computational advantages, so the MAP-receiver can maximize the function

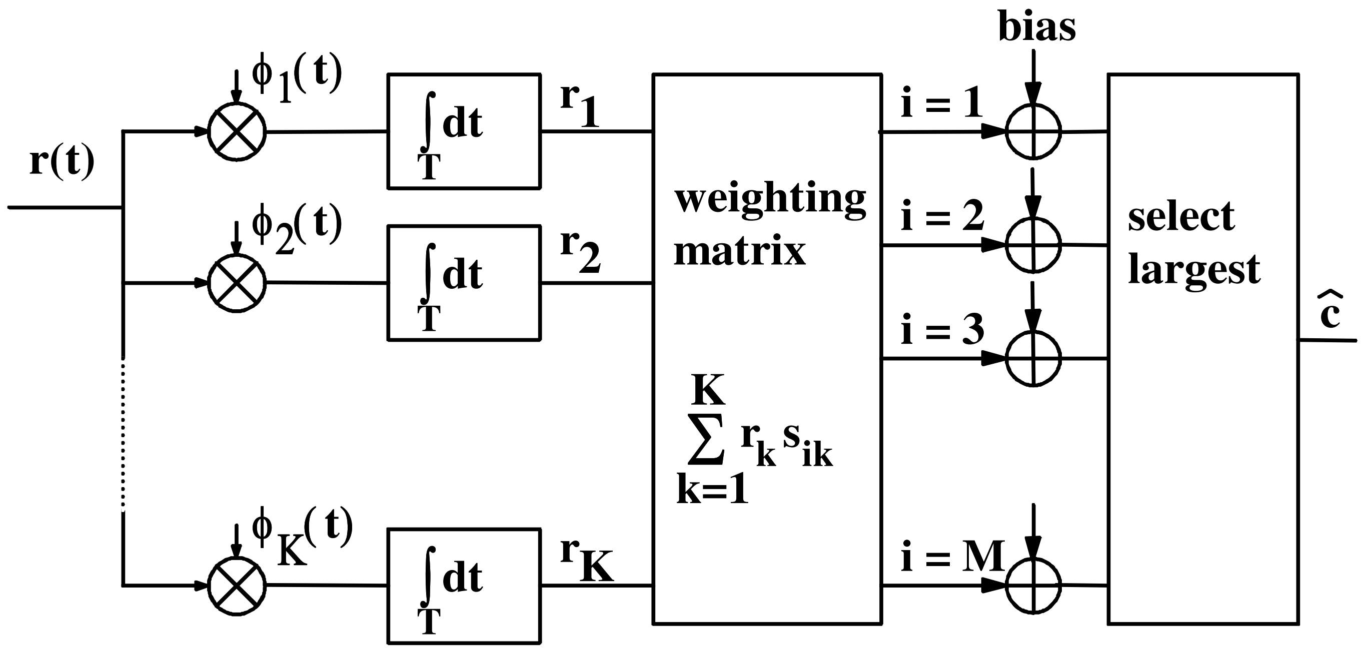
$$\ln\left(P\left(c_i\right)p\left(R\middle|c_k\right)\right)=\ln P\left(c_i\right)-\frac{K}{2}\ln\left(\pi N_o\right)-\frac{1}{N_o}\sum_{k=1}^{K}\left(r_k-s_{ik}\right)^2$$

$$=\ln P\left(c_i\right)-\frac{K}{2}\ln\left(\pi N_o\right)-\frac{1}{N_o}\sum_{k=1}^{K}r_k^2+\frac{2}{N_o}\sum_{k=1}^{K}r_k s_{ik}-\frac{1}{N_o}\sum_{k=1}^{K}s_{ik}^2 \qquad (29)$$

Only terms containing the index *i* need to be considered in the MAP-receiver. It is also easy to show that the last sum represents the energy of the received pulse waveforms. Therefore the MAP decision rule in the AWGN-channel can be written as

$$\hat{c} = c_i \rightarrow \underset{i}{\mathrm{MAX}} \left\{ \sum_{k=1}^{K} r_k s_{ik} + \frac{N_o}{2} \ln P(c_i) - \frac{E_{s_i}}{2} \right\} \quad (30)$$

Block diagram of the correlation receiver according to the equation above



The optimum receiver can also be implemented in an alternative way. The response to r(t) of a filter with the impulse response $h_k(t) = \phi_k(T - t)$ **is**

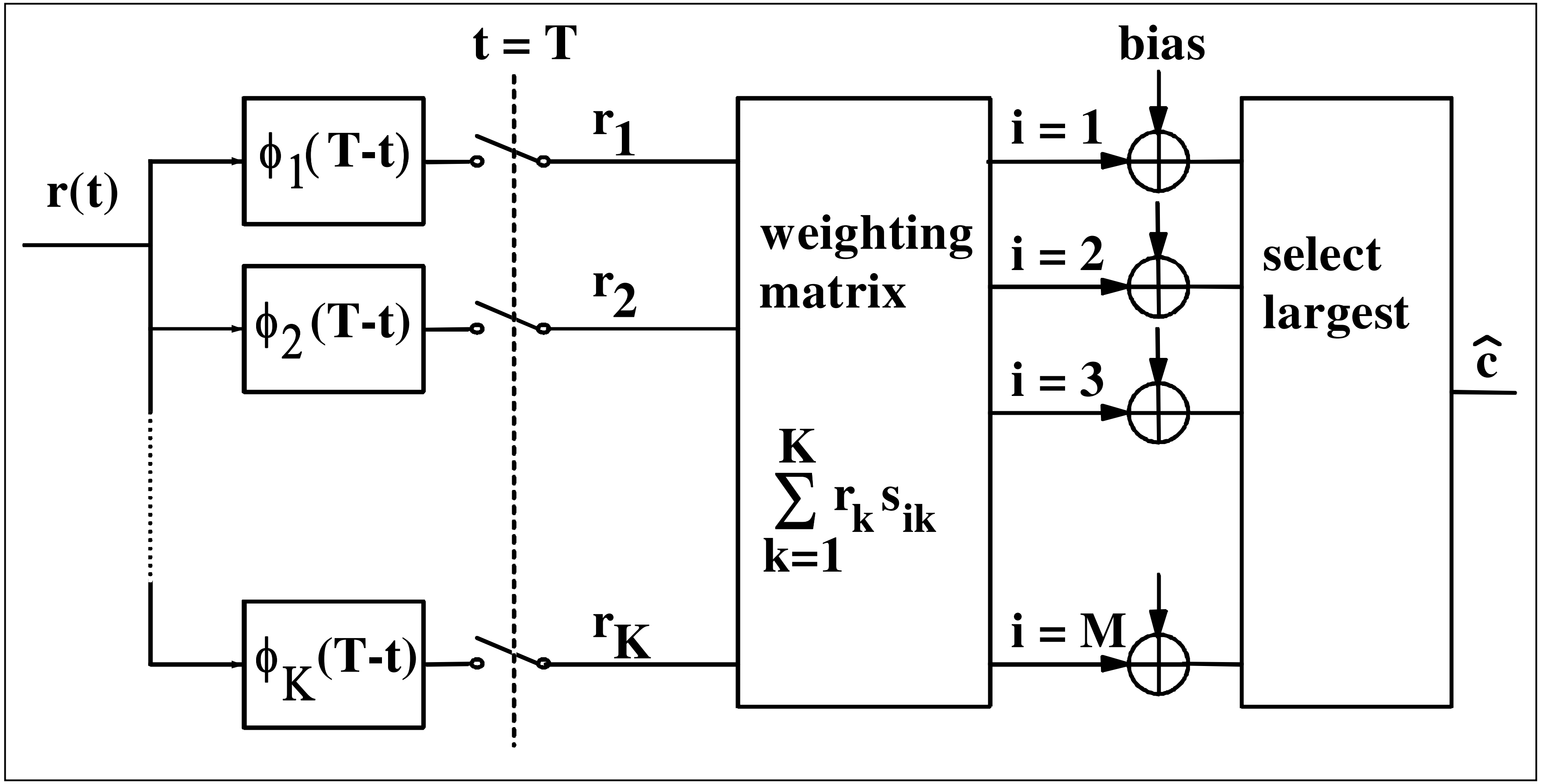
$$y(t) = \int_{-\infty}^{\infty} r(u) h_k(t-u) du = \int_{-\infty}^{\infty} r(u) \phi_k(T - t + u) du \qquad \textbf{(31)}$$

On the time instant t = T the filter output is

$$y(T) = \int_{-\infty}^{\infty} r(u) \phi_k(u) du = r_k \qquad \textbf{(32)}$$

so this so called matched filter gives the same output as the corresponding correlator in the correlator receiver. The block diagram of the matched filter receiver is shown in the figure below.

Both the correlator receiver and matched filter receiver are optimum single symbol receivers in the AWGN-channel in the sense that they minimize the symbol error probability. In an arbitrary channel the correlator receiver will make decision according to the least error energy $\int (r(t) - s_i(t))^2 \, dt$, while the matched filter receiver will maximize SNR at the decision time instant, but they don't necessarily minimize SEP.



EXAMPLE 2. Optimum receiver for binary antipodal signalling and its performance in the AWGN-channel

In binary antipodal signalling the symbol values and their probabilities are

$$c_1 = -1, \quad P\{c_k = -1\} = P_1 = 0,5$$
$$c_2 = +1, \quad P\{c_k = +1\} = P_2 = 0,5 \qquad \textbf{(33a,b)}$$

The received pulse waveforms are thus

$$s_1(t) = -x(t)$$
$$s_2(t) = x(t) \qquad \textbf{(34a,b)}$$

The choice of base functions is straightforward, only one function is needed

$$\phi(t) = \frac{1}{\sqrt{E_x}} x(t) \qquad \textbf{(35)}$$

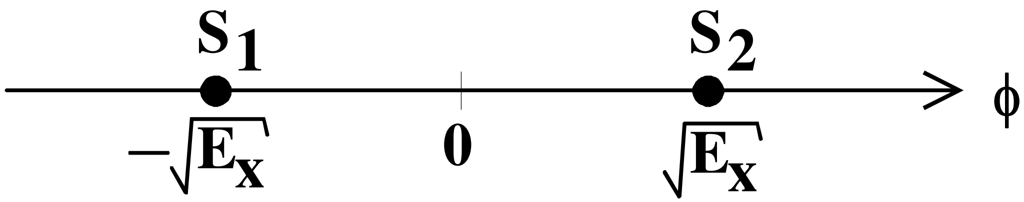
Each signal vector comprises one component:

$$s_1 = \int_T -x(t)\phi(t)dt = -\int_T x(t)\frac{1}{\sqrt{E_x}}x(t)dt = -\frac{1}{\sqrt{E_x}}\int_T x^2(t)dt = -\sqrt{E_x}$$

$$s_2 = \int_T x(t)\phi(t)dt = \int_T x(t)\frac{1}{\sqrt{E_x}}x(t)dt = \frac{1}{\sqrt{E_x}}\int_T x^2(t)dt = \sqrt{E_x}$$

(36a,b)

The signal vector ends define a geometrical pattern which is called the signal constellation. In the actual case the constellation points are on a straight line, we have a one-dimensional signal geometry.



The threshold *u* between the decision areas is solved from the equality

$$P(c=-1)\,p_n(u-s_1) = P(c=+1)\,p_n(u-s_2) \qquad \textbf{(37)}$$

In the general case this equation can be written as

$$P_1 \frac{1}{\sqrt{\pi N_o}} e^{-\frac{(u-s_1)^2}{N_o}} = P_2 \frac{1}{\sqrt{\pi N_o}} e^{-\frac{(u-s_2)^2}{N_o}} \qquad \textbf{(38)}$$

Taking the logarithm of both sides one gets

$$\ln P_1 - \frac{(u-s_1)^2}{N_o} = \ln P_2 - \frac{(u-s_2)^2}{N_o} \qquad \textbf{(39)}$$

from which *u* can be solved:

$$u = s_2 + s_1 + \frac{N_o}{2(s_2 - s_1)} \ln\left(\frac{P_1}{P_2}\right) = \frac{N_o}{4\sqrt{E_x}} \ln\left(\frac{P_1}{P_2}\right), \text{ when } s_2 = -s_1 = \sqrt{E_x} \qquad \textbf{(40)}$$

The former expression is valid for any one-dimensional constellation, while the latter expression is valid for antipodal signalling. With equiprobable symbol the threshold value $u = 0$.

Symbol error probability

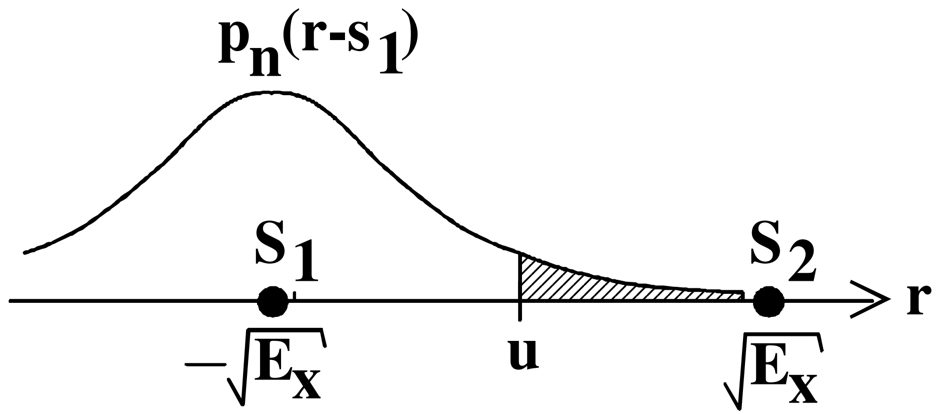
When the symbol c_1 is transmitted a decision error is made if r > u, and when the symbol c_2 is transmitted a decision error is made if r < u

The conditional symbol error probabilities are given by the surfaces under the density function

$$P\left(e|c=-1\right)=\int_{u}^{\infty} p_n\left(r-s_1\right)dr$$

(41a,b)

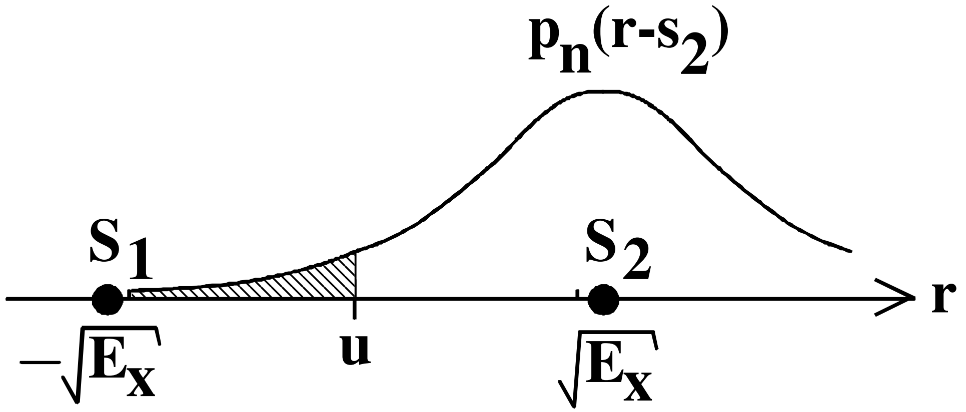
$$P\left(e|c=+1\right)=\int_{-\infty}^{u} p_n\left(r-s_2\right)dr$$



Integration of the first conditional probability gives:

$$P\left(e|c=1\right)=\int_{u}^{\infty}\frac{1}{\sqrt{\pi N_o}}e^{-\frac{\left(r-s_1\right)^2}{N_o}}dr$$

(42)



Changing the integration variable $x = \frac{(r - s_1)\sqrt{2}}{\sqrt{N_o}}$ **gives**

$$P(e|c = -1) = \int_{\frac{u - s_1}{\sqrt{N_o/2}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = Q\left(\frac{u - s_1}{\sqrt{N_o/2}}\right) \quad (43)$$

where Q(x) is the complement of the normal cumulative distribution function, which is a tabulated function.

The second conditional error probability can be determined in the same manner, but can also be obtained more directly by utilizing the symmetry of the noise density function.
The surface under the left density function tail is the same as the surface of a right density function tail starting at the same distance from the mean value. Therefore

$$P(e|c = 1) = Q\left(-\frac{u - s_2}{\sqrt{N_o/2}}\right) = Q\left(\frac{s_2 - u}{\sqrt{N_o/2}}\right) \quad (44)$$

In both cases the argument of the Q-function is the ratio between the distance from the constellation point to the decision area border and the r.m.s. noise. This can be generalized also to other cases than the optimum receiver.

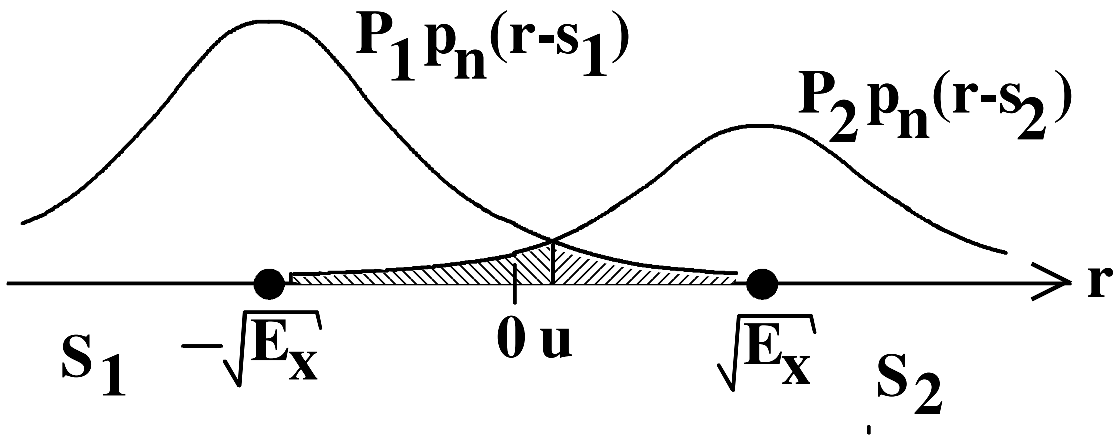
The general expression of the symbol error probability is

$$P(e) = P(-1)P\left(e \middle| c = -1\right) + P(1)P\left(e \middle| c = 1\right) = P(-1)Q\left(\frac{u - s_1}{\sigma_n}\right) + P(1)Q\left(\frac{s_2 - u}{\sigma_n}\right) \quad (45)$$

Insertion of the actual values gives

$$P(e) = \frac{1}{2}Q\left(\frac{0-\left(-\sqrt{E_x}\right)}{\sqrt{N_o/2}}\right) + \frac{1}{2}Q\left(\frac{\sqrt{E_x}-0}{\sqrt{N_o/2}}\right) = Q\left(\sqrt{\frac{2E_x}{N_o}}\right) \quad (46)$$

With unequal symbol probabilities the MAP decision threshold is moved and the BEP will change.



MPAM-signalling and its performance in the AWGN-channel

MPAM = M-ary Pulse Amplitude Modulation

In single symbol transmission the received MPAM-signal is

$$r(t) = cx(t) + n(t) \qquad (47)$$

where c obtains the values ±1, ±3, ±5,...±(M-1). Each value occures with the same probability: $P(c_i) = 1/M$.

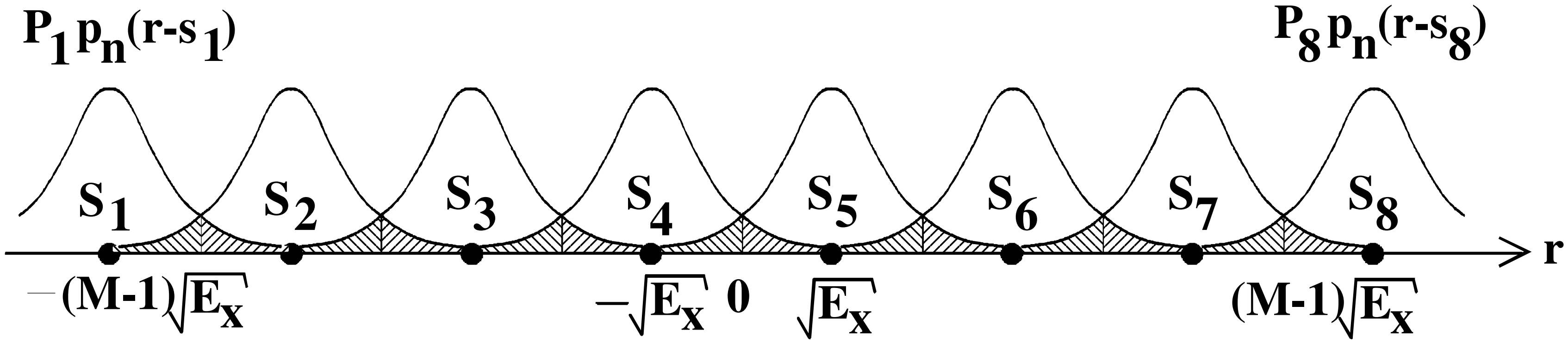
The choice of orthonormal functions is simple, only one function is needed

$$\phi(t) = \frac{1}{\sqrt{E_x}} x(t) \qquad (48)$$

Each signal vector comprises one component:

$$s_i = \int_T x(t)\phi(t)dt = \int_T cx(t)\frac{1}{\sqrt{E_x}} x(t)dt = \frac{c}{\sqrt{E_x}} \int_T x^2(t)dt = c\sqrt{E_x} \qquad (49)$$

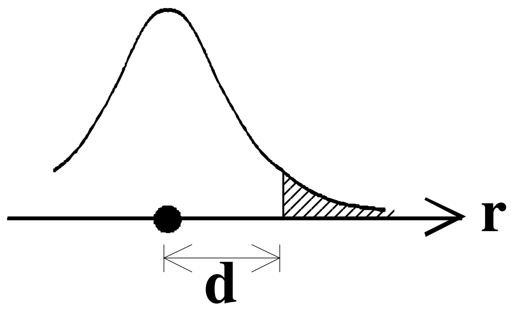
The constellation of 8PAM and the density functions conditioned on the transmitted symbol value are given in the figure below.



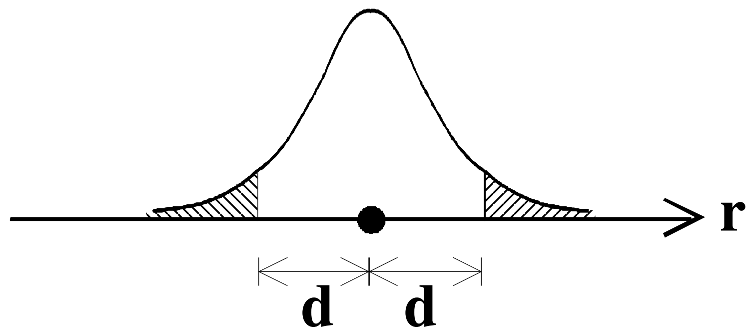
With equally probable symbol values the borders between decision areas are halfway between the constellation points.

Average symbol error probability:

Two cases: 2 edge symbols **M-2 inner symbols**



The conditional symbol error



probability of the edge symbols:

$$P\left(e \middle| edge\ symbol\right) = Q\left(\frac{d}{\sigma_n}\right) \qquad (50)$$

The conditional symbol error probability of the inner symbols:

$$P\left(e \middle| inner\ symbol\right) = 2Q\left(\frac{d}{\sigma_n}\right) \qquad \mathbf{(51)}$$

The average symbol error probability:

$$P(e) = \frac{1}{M}\left[2 \cdot Q\left(\frac{d}{\sigma_n}\right) + (M-2) \cdot 2Q\left(\frac{d}{\sigma_n}\right)\right] = \frac{2}{M}(M-2)Q\left(\frac{d}{\sigma_n}\right) = 2\left(1 - \frac{1}{M}\right)Q\left(\sqrt{\frac{2E_x}{N_o}}\right)$$

(52)

The last version is valid for the optimum receiver in the AWGN-channel, whereas the second last version is valid for any ideal channel containing additive Gaussian noise.

It is more usual to express SEP as function of average received energy. The average energy can be derived from the pulse energy in the following way:

$$E_s = \sum_{i=1}^{M} P(c_i) E_{c_i} = \frac{2}{M} \sum_{i=1}^{M/2} (2i-1)^2 E_x = \frac{2E_x}{M}\left[4\sum_{i=1}^{M/2} i^2 - 4\sum_{i=1}^{M/2} i - \sum_{i=1}^{M/2} 1\right]$$

$$= \frac{2E_x}{M}\left[4\left(\frac{1}{6}\frac{M}{2}\left(\frac{M}{2}+1\right)\left(2\frac{M}{2}+1\right)\right) - 4\frac{1}{2}\frac{M}{2}\left(\frac{M}{2}+1\right) + \frac{M}{2}\right] = \frac{E_x}{3}\left[M^2-1\right] \quad (53)$$

This leads to the following SEP-expression:

$$P(e) = 2\left(1-\frac{1}{M}\right) Q\left(\sqrt{\frac{3}{M^2-1}\cdot\frac{2E_s}{N_o}}\right) \quad (54)$$

Sometimes SEP is presented as a function of the average bit energy. Because

$$E_b = \frac{E_s}{\log_2 M} \rightarrow P(e) = 2\left(1-\frac{1}{M}\right) Q\left(\sqrt{\frac{3\cdot\log_2 M}{M^2-1}\cdot\frac{2E_b}{N_o}}\right) \quad (55)$$

Further one can represent SEP as function of maximum symbol energy.

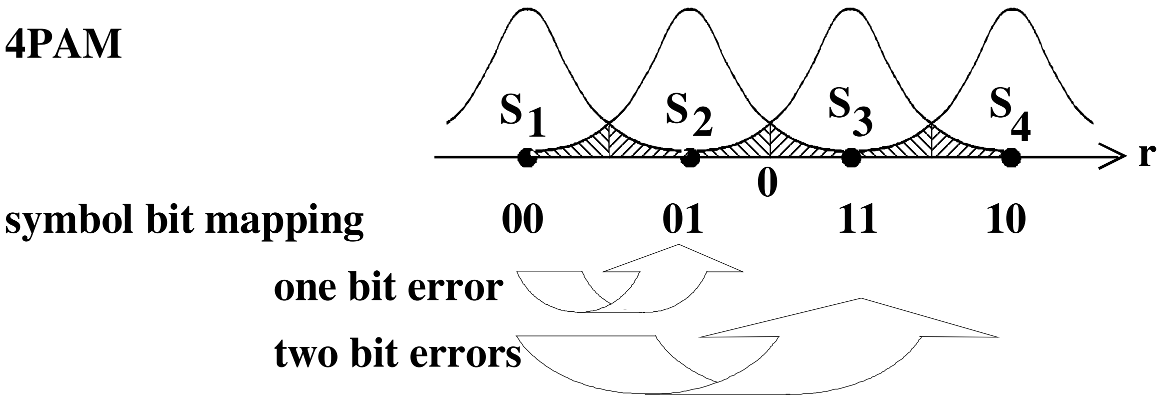
Bit error probability

Transmission performance requirements are often defined in terms of bit error probability (BEP). BEP will depend on how bits are mapped into symbols.

When a symbol error occurs at least one bit is erroneous, but in the worst case all bits can be erroneous.

It is likely that a symbol error will lead to a symbol in an adjacent decision area. Therefore the bits should be mapped into symbols in such a way that the symbols in adjacent decision areas differ only in one bit. In MPAM-systems this is achieved with Gray-coding.

However, it is possible that an erroneous decision gives a symbol in a decision area outside the adjacent areas to the correct decision area. Especially on high BEP-values a more exact analysis is needed to get an good BEP-estimate.



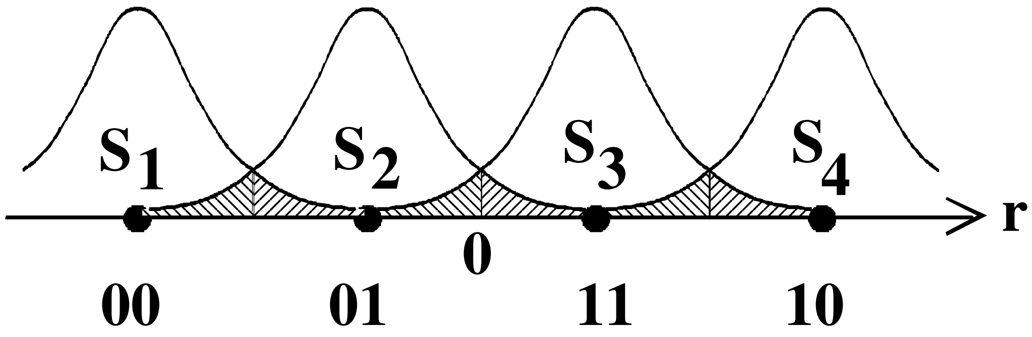
THE GRAY CODER

Coder input sequence: $\{c_1, c_2, ..., c_K\}, \quad 2^K = M$

Coder output sequence: $\{c_1, c_1 \oplus c_2, c_2 \oplus c_3, ..., c_{K-1} \oplus c_K\}, \quad K\ bits$

0000	Input	$\oplus$ stands for modulo-2	0000	Output
0001	sequence	summation (exclusive or)	0001	sequence
0010	16PAM	logic circuit	0011	
0011			0010	
0100			0110	
0101			0111	
0110			0101	
0111	→	0, 0⊕1, 1⊕1, 1⊕0 →	0100	
1000			1100	
1001			1101	
1010			1111	
1011			1110	
1100			1010	
1101			1011	
1110			1001	
1111			1000	

Example 3: BEP of a Gray-coded 4PAM-system



First we look for eventual symmetries by analyzing the number of bit errors for different symbol errors.

symbol	distance in decision areas of error symbol			
	1	2	3	
00	1	2	1	
01	1,1	2		number of bit errors
11	1,1	2		
10	1	2	1	

Because of symmetry only the two first symbols need to be investigated

Now the conditional bit error probabilities are:

$$BEP_{00} = \frac{1}{2}P(00 \to 01) + \frac{2}{2}P(00 \to 11) + \frac{1}{2}P(00 \to 10)$$

$$= 0,5P\{d < n < 3d\} + P\{3d < n < 5d\} + 0,5P\{n > 5d\}$$

$$= 0,5\left[Q\left(\frac{d}{\sigma}\right) - Q\left(\frac{3d}{\sigma}\right)\right] + \left[Q\left(\frac{3d}{\sigma}\right) - Q\left(\frac{5d}{\sigma}\right)\right] + 0,5Q\left(\frac{5d}{\sigma}\right)$$

$$= 0,5Q\left(\frac{d}{\sigma}\right) + 0,5Q\left(\frac{3d}{\sigma}\right) - 0,5Q\left(\frac{5d}{\sigma}\right)$$

$$BEP_{01} = \frac{1}{2}P(01 \to 00) + \frac{2}{2}P(01 \to 11) + \frac{2}{2}P(01 \to 10)$$

$$= 0,5P\{n < -d\} + 0,5P\{d < n < 3d\} + P\{n > 3d\}$$

$$= 0,5Q\left(\frac{d}{\sigma}\right) + 0,5\left[Q\left(\frac{d}{\sigma}\right) - Q\left(\frac{3d}{\sigma}\right)\right] + Q\left(\frac{3d}{\sigma}\right) = Q\left(\frac{d}{\sigma}\right) + 0,5Q\left(\frac{3d}{\sigma}\right)$$

The average BEP is

$$BEP = \frac{1}{M/2}\left[0,5Q\left(\frac{d}{\sigma}\right)+0,5Q\left(\frac{3d}{\sigma}\right)-0,5Q\left(\frac{5d}{\sigma}\right)+Q\left(\frac{d}{\sigma}\right)+0,5Q\left(\frac{3d}{\sigma}\right)\right]$$

$$=\frac{1}{2}\left[1,5Q\left(\frac{d}{\sigma}\right)+Q\left(\frac{3d}{\sigma}\right)-0,5Q\left(\frac{5d}{\sigma}\right)\right]\approx 0,75Q\left(\frac{d}{\sigma}\right)$$

The last part of the expression is an approximate value which is less than 1% in error if BEP < 0,1.

Generally, $BEP = \frac{1}{\log_2 M} SEP$ **for low BEP-values, and in MPAM is then**

$$BEP = \frac{2}{\log_2 M}\left(1-\frac{1}{M}\right)Q\left(\sqrt{\frac{3\cdot\log_2 M}{M^2-1}\cdot\frac{2E_b}{N_o}}\right)=\frac{2}{\log_2 M}\left(1-\frac{1}{M}\right)Q\left(\sqrt{\frac{3}{M^2-1}\cdot\frac{2E_s}{N_o}}\right) \quad (56)$$

Table 1. $10\log(E_b/N_o)$-values and bandwidths needed for different BEP-values with the optimum receiver when the bit rate is constant

M	BEP = 10^{-3}	BEP = 10^{-6}	BEP = 10^{-9}	spectrum efficiency $\eta = \frac{R_b}{B} \leq 2\log_2 M$
2	6.8 dB	10.5 dB	12.6 dB	2
4	7.6	11.4	13.5	4
8	11.8	15.8	17.9	6
16	16.4	20.5	22.6	8
32	21.3	25.5	27.7	10
64	26.3	30.7	32.9	12
128	31.5	36.0	38.2	14
256	36.8	41.4	43.6	16

ORTHOGONAL signalling and its performance in the AWGN-channel

In orthogonal single symbol transmission the received signal is

$$r(t) = s_i(t) + n(t) \qquad (57)$$

where i = 1, 2, 3,..., M, and each pulse waveform is orthogonal with respect to any other pulse waveform. We will assume that each waveform (or the corresponding symbol occures with the same probability: P(c_i) = 1/M.

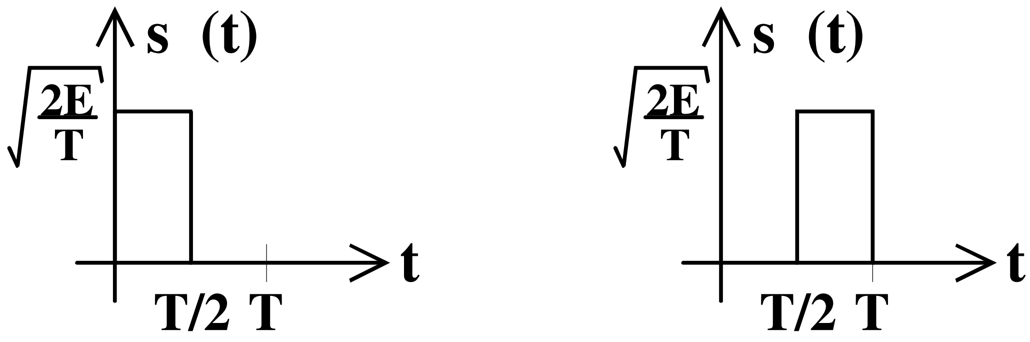
The choice of basis functions is straightforward, because the pulse waveforms are orthogonal they form an orthonormal set after normalization:

$$\phi_i(t) = \frac{1}{\sqrt{E_{s_i}}} s_i(t), \quad i = 1, 2, 3, \ldots, M \qquad (58)$$

The dimension of the signal vector equals the number of symbols, but each M-dimensional signal vector comprises only one component differing from zero. The values of the components are simply $s_{ik} = \sqrt{E_{s_i}}\,\delta_{ik}$

Example 4: Orthogonal PPM

A simple set of orthogonal function are time-shifted rectangular pulses. In the binary case we get the pulse waveform shown in the figure below.



The orthonormal functions are

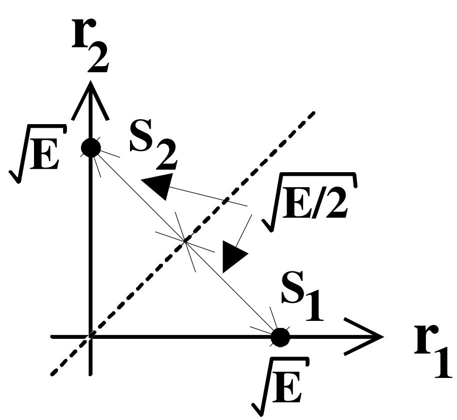
$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_{s_1}}} = \sqrt{\frac{2}{T}}\,\mathrm{rect}\left(\frac{t - T/4}{T/2}\right)$$

$$\phi_2(t) = \frac{s_2(t)}{\sqrt{E_{s_2}}} = \sqrt{\frac{2}{T}}\,\mathrm{rect}\left(\frac{t - 3T/4}{T/2}\right)$$

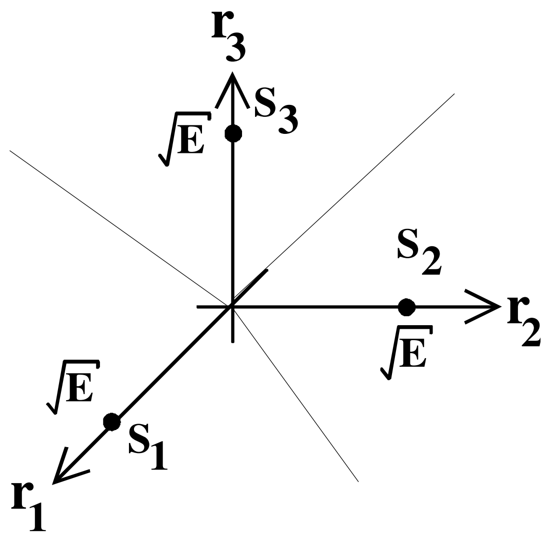
The signal vectors are

$$S_1 = \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix} = \begin{bmatrix} \int s_1(t)\phi_1(t)dt \\ \int s_1(t)\phi_2(t)dt \end{bmatrix} = \begin{bmatrix} \sqrt{E} \\ 0 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix} = \begin{bmatrix} \int s_2(t)\phi_1(t)dt \\ \int s_2(t)\phi_2(t)dt \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{E} \end{bmatrix}$$



The example can be easily generalized, but a graphical presentation of the orthogonal constellation is difficult when M > 3.



Average symbol error probability

An exact closed-form SEP-expression cannot be given, because the decision areas are not right-angled, which is necessary with Gaussian noise. An exception is the binary case, where the optimum receiver SEP is:

$$SEP = Q\left(\frac{d}{\sigma}\right) = Q\left(\frac{\sqrt{E/2}}{\sqrt{N_o/2}}\right) = Q\left(\sqrt{\frac{E}{N_o}}\right) \quad (59)$$

An SEP-expression in integral form for the general M-orthogonal optimum system can be obtained in the following way.

A symbol error occurs if the distance between the received vector and an other constellation point is shorter than the distance to the constellation point corrresponding to the transmitted symbol.

The event of a correct decision can be defined as

$$\{c|i\} = \{|R - S_i| < |R - S_j| \,|\, i\} \quad \forall j \neq i \quad (60)$$

On the other hand is

$$\left|R - S_j\right|^2 = \left|R\right|^2 + \left|S_j\right|^2 - 2R \cdot S_j = \left|R\right|^2 + E_j - 2r_j\sqrt{E_j} \qquad \textbf{(61)}$$

If we assume that all pulse waveforms have equal energy, the event of a correct decision can be simplified to

$$\{c\} = \{r_j < r_i\} \quad \forall j \neq i \qquad \textbf{(62)}$$

When the symbol S_i is transmitted the received signal vector components are

$$S_i \ \textit{transmitted} \rightarrow r_j = \begin{cases} \sqrt{E} + n_i = \alpha, \ j = i \\ n_j, \ j \neq i \end{cases} \qquad \textbf{(63)}$$

The correct decision probability conditioned on a certain symbol and a given vector component value is:

$$P\left(c \,|\, i, r_i = \alpha\right) = P\left(n_1 < \alpha, ..., n_{i-1} < \alpha, n_{i+1} < \alpha, ..., n_M < \alpha\right) \qquad \textbf{(64)}$$

Due to the statistical independence and equal distributions of the noise components this conditional probability can be rewritten as:

$$P\left(c\,|\,i, r_i = \alpha\right) = \left[P\left(n < \alpha\right)\right]^{M-1} = \left[\int_{-\infty}^{\alpha} p_n\left(x\right)dx\right]^{M-1} = \left[1 - Q\left(\sqrt{\frac{2\alpha^2}{N_o}}\right)\right]^{M-1} \tag{65}$$

Because of the symmetry of the constellation the conditionl probability doesn't depend on the transmitted symbol *i* and after averaging over *a* and subtraction from 1 we get the average symbol error probability:

$$\begin{aligned} SEP &= 1 - \int_{-\infty}^{\infty} p_n\left(\alpha - \sqrt{E}\right) P\left(C\,|\,\alpha\right) d\alpha \\ &= 1 - \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi N_o}} e^{-\frac{\left(\alpha - \sqrt{E}\right)^2}{N_o}} \left[1 - Q\left(\sqrt{\frac{2\alpha^2}{N_o}}\right)\right]^{M-1} d\alpha \end{aligned} \tag{66}$$

For other cases than *M* = 2 this must be numerically computed.

SEP union upper bound

A symbol error will occur if any of the other noise components are larger than the component corresponding to the transmitted symbol or

$$P\left(e\,|\,i, r_i = \alpha\right) = P\left(n_1 > \alpha \cup \ldots \cup n_{i-1} > \alpha \cup n_{i+1} > \alpha \cup \ldots \cup n_M > \alpha\right) \quad \textbf{(67)}$$

The probability of a union cannot easily be determined, but application of the union bound gives a simple upper bouind.

From probability theory it is known that

$$P\left(A_1 \cup A_2 \cup A_3 \cup \ldots\right) \le P\left(A_1\right) + P\left(A_2\right) + P\left(A_3\right) + \ldots \quad \textbf{(68)}$$

Application of the union bound gives the following upper bound for the conditional SEP:

$$P\left(e\,|\,i, r_i = \alpha\right) \le \sum_{\substack{j=1 \\ j \ne i}}^{M} P\left(n_i > \alpha\right) = (M-1)Q\left(\sqrt{\frac{2\alpha^2}{N_o}}\right) \quad \textbf{(69)}$$

Averaging over the symbols and α-values gives

$$SEP \leq (M-1)\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi N_o}} e^{-\frac{\left(\alpha-\sqrt{E}\right)^2}{N_o}} Q\left(\sqrt{\frac{2\alpha^2}{N_o}}\right) d\alpha \qquad (70)$$

A comparison of this integral with the integral in the conditional probability of a correct decision shows that this integral equals that integral for the binary case. But this probability was derived sepatately and is known, so the upper bound can be expressed as

$$SEP \leq (M-1) Q\left(\sqrt{\frac{E}{N_o}}\right) \qquad (71)$$

This upper bound will be quite tight on low SEP-values.

The average bit error probability

We assume the symbols to be equiprobable and that all pulse waveforms have the same energy.

Due to the nature of orthogonal signalling the probability to be in a certain erraneous decision area in a symbol error situation is the same for all error symbols. Therefore bit mapping into symbols is not an important issue, any mapping will give the same bit error performance.

The probability of being in a certain decision area, given a symbol error is 1/(M-1).

The number of decision areas where one gets N bit errors is given by the binomial factor $\binom{K}{N}$, where $K = \log_2 M$ is the number of bits in a symbol.

The average bit error probability is

$$BEP = \sum_{N=1}^{K} \begin{Bmatrix} \textit{fraction of} \\ \textit{error bits} \end{Bmatrix} \begin{Bmatrix} \textit{number of decision} \\ \textit{areas with N bit errors} \end{Bmatrix} \begin{Bmatrix} \textit{probability of being in} \\ \textit{a given decision area} \end{Bmatrix}$$

$$= \sum_{N=1}^{K} \frac{N}{K} \binom{K}{N} \frac{SEP}{2^K - 1} = \frac{SEP}{\left(2^K - 1\right)} \sum_{N=1}^{K} \frac{N \cdot K!}{K \cdot N!(K-N)!}$$

$$= \frac{SEP}{\left(2^K - 1\right)} \sum_{N=1}^{K} \frac{(K-1)!}{(N-1)!((K-1)-(N-1))!} = \frac{SEP}{\left(2^K - 1\right)} \sum_{N=1}^{K} \binom{K-1}{N-1} \tag{72}$$

The sum of of binomial factors is 2^{K-1}, which gives average BEP:

$$BEP = \frac{2^{K-1}}{2^K - 1} SEP = \frac{M/2}{M-1} SEP \tag{73}$$

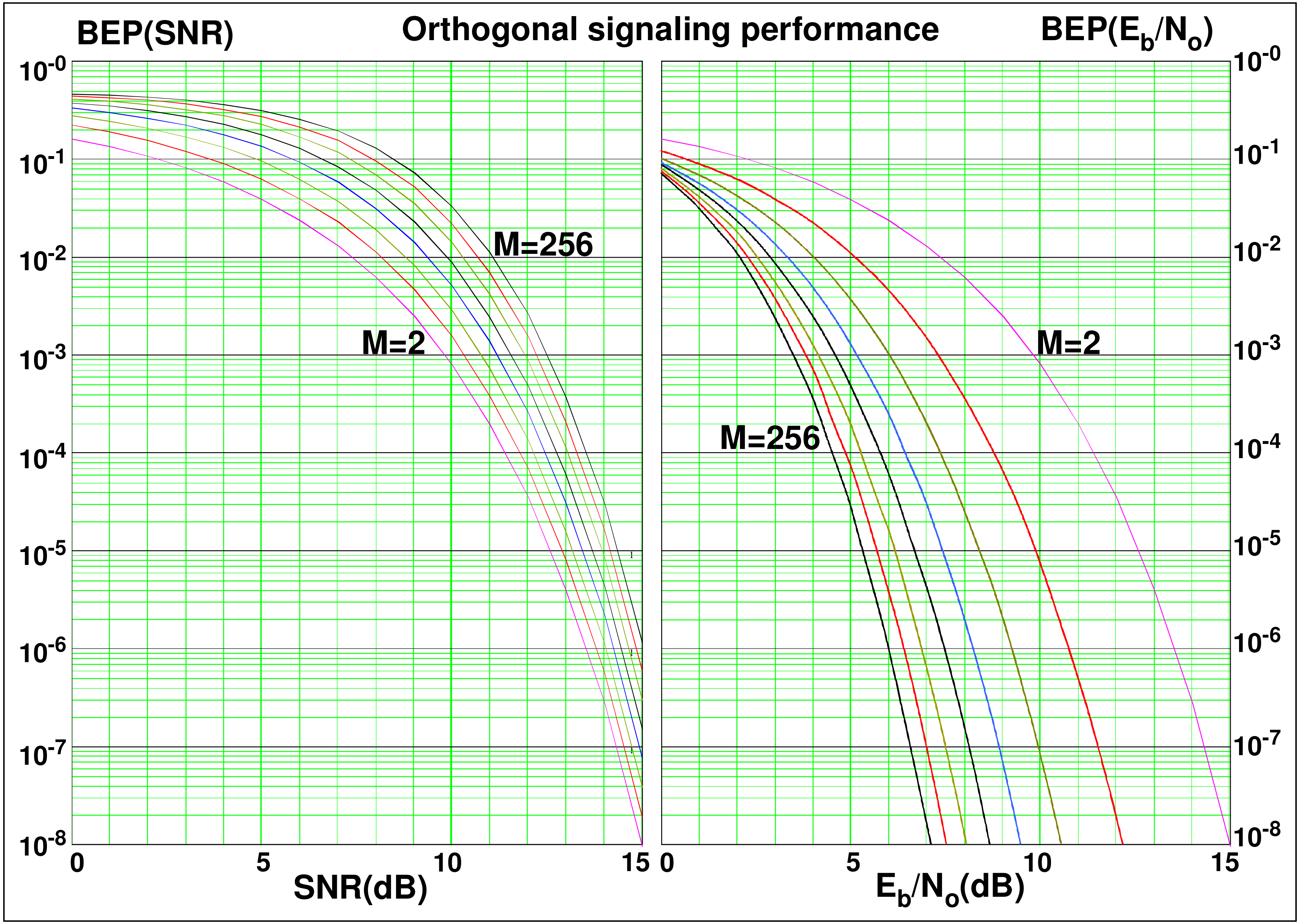
Using the SEP union bound:

$$BEP \le \frac{M/2}{M-1}(M-1) Q\left(\sqrt{\frac{E}{N_o}}\right) = \frac{M}{2} Q\left(\sqrt{\frac{E}{N_o}}\right) = \frac{M}{2} Q\left(\sqrt{\frac{E_b \log_2 M}{N_o}}\right) \tag{74}$$

Table 2. $10\log(E_b/N_o)$-values and bandwidths are needed for different BEP-values with the optimum receiver with constant bit rate, and BW being the bandwidth of binary transmission.

M	BEP = 10^{-3}	BEP = 10^{-6}	BEP = 10^{-9}	spectrum efficiency $\eta = \frac{R_b}{B} \le \frac{2\log_2 M}{M}$
2	9.8 dB	13.5 dB	15.6 dB	1.00
4	7.3	10.8	12.7	1.00
8	6.1	9.3	11.1	0.75
16	5.3	8.2	10.0	0.50
32	4.7	7.5	9.2	0.3125
64	4.3	6.9	8.5	0.1875
128	3.9	6.4	8.0	0.109375
256	3.7	6.0	7.5	0.0625

A comparison with the corresponding table for MPAM shows the radically different natures of MPAM and M:ary orthogonal signalling. The former is suitable for use in situations with bandwidth constraints but no power constraints while the latter is suitable for use in channels with power constraints but no bandwidth constraints.



Orthogonal signaling performance
BEP(SNR)
BEP(E_b/N_o)
10^-0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
10^-8
M=256
M=2
M=2
M=256
0
5
10
15
SNR(dB)
E_b/N_o(dB)