# S.72.333 Post-graduate Seminar in Radio Communications

# Single Symbol Optimum Receiver Principles Part II

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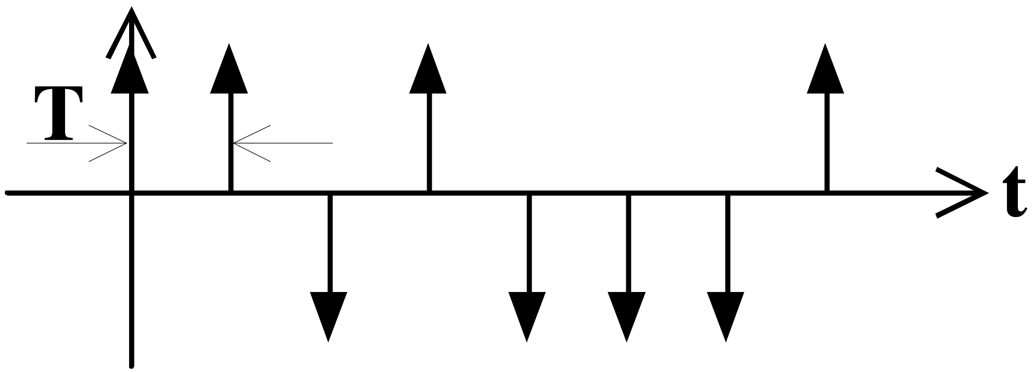
# SYMBOL SEQUENCE RECEIVER IN THE AWGN-CHANNEL WITH SYMBOL-BY-SYMBOL DECISION

**The symbol sequence source generates N subsequent symbols with the symbol rate $R_S = 1/T$ e.g. a bipolar binary sequence:**

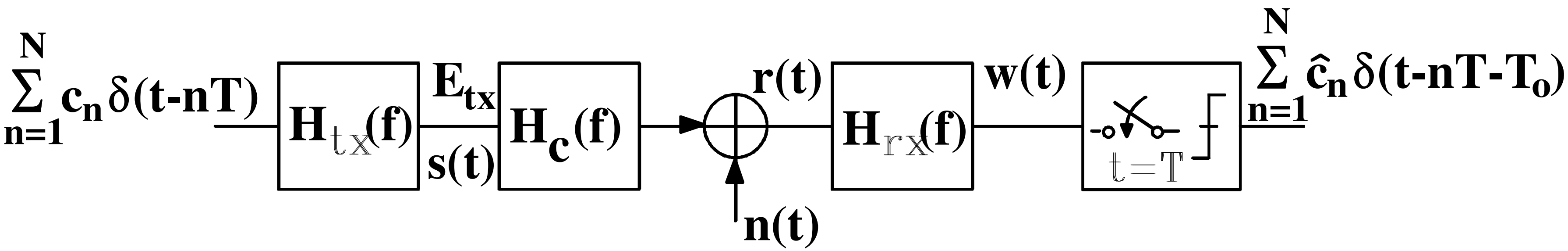



**The sequence can have either a constraint length or be infinitely long.**

**If there are no bandwidth constraints in the transmission channel one can use pulse waveforms limited in duration to one symbol interval or shorter (full response signalling). With proper decision sampling timing, the subsequent symbols can be decided independently and optimally with the single-symbol optimum receiver and the error performance is the same as in single symbol reception.**

**In a bandlimited channel the received pulse is no more limited to a symbol interval and after receiver filtering there will be interaction from neighbour symbols in the decision.**

## System model




## Signal analysis

## Source signal:

$$s(t) = \left( \sum_{n=1}^{N} c_n \delta\left(t - nT\right) \right) \otimes h_{tx}(t) = \sum_{n=1}^{N} c_n h_{tx}\left(t - nT\right) \qquad (75)$$

## Received signal:

$$r(t) = \left( \sum_{n=1}^{N} c_n h_{tx}\left(t - nT\right) \right) \otimes h_c(t) + n(t) = \sum_{n=1}^{N} c_n y\left(t - nT\right) + n(t) \qquad (76)$$

**Decision signal**

$$w(t) = \left( \sum_{n=1}^{N} c_n x\left(t - nT\right) \right) \otimes h_{rx}(t) + n(t) \otimes h_{rx}(t)$$
$$= \sum_{n=1}^{N} c_n z\left(t - nT\right) + n_{rx}(t) \qquad (77)$$

**The operator "⊗" denotes convolution:**

$$f(t) \otimes g(t) = \int_{-\infty}^{\infty} f(u) g(t-u) du = \int_{-\infty}^{\infty} f(t-u) g(u) du \qquad (78)$$

**The decision circuit input pulse waveform is resulting from the overall linear filtering:**

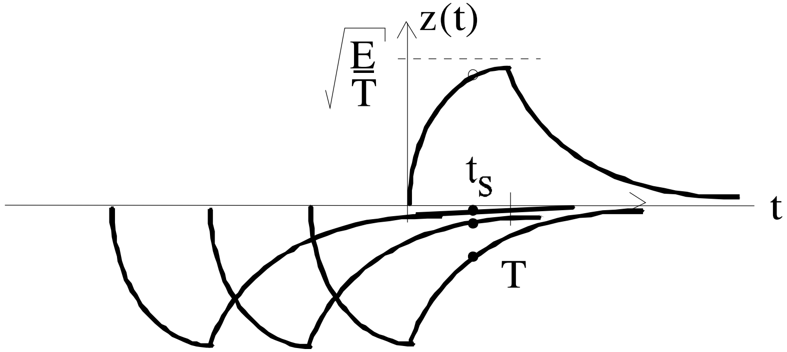
$$z(t) = h_{rx}(t) \otimes h_c(t) \otimes h_{tx}(t) = \mathrm{F}^{-1}\left\{ H_{rx}(f) H_c(f) H_{tx}(f) \right\} \qquad (79)$$

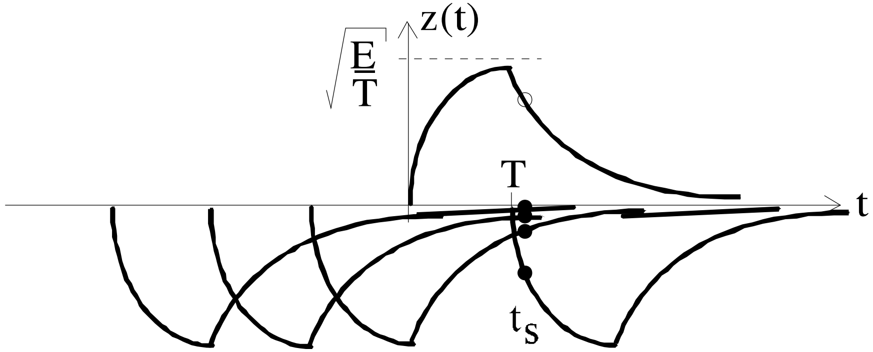
**When decision sampling takes place at the instants $t_s + kT$, the decision sample can be written as**

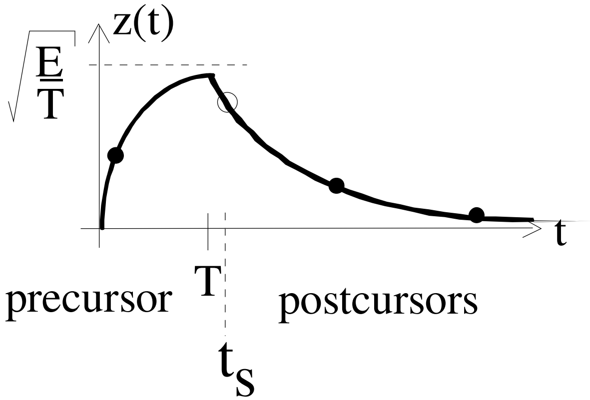
$$w(t_s + kT) = w_k = \sum_{n=1}^{N} c_n z\left(t_s + (k-n)T\right) + n_{rx}(t_s + (k-n)T)$$
$$= c_k z_0 + \sum_{\substack{n=1 \\ n \neq k}}^{N} c_n z_{k-n} + n_{rx} \qquad (80)$$

**In the last expression the first term is the desired signal sample, the second term represents the disturbance produced by the other symbol when the receiver makes its decision. This sum term is called InterSymbol Interference (ISI). The third term is the Gaussian noise sample.**

**Graphical visualization of decision signal and samples in a binary receiver in a bandlimited Gaussian channel.**




**All samples can be obtained from the basic decision pulse waveform.**

# ERROR PERFORMANCE OF THE SYMBOL-BY-SYMBOL RECEIVER IN THE BANDLIMITED AWGN-CHANNEL

**ISI is a random signal which disturbs the decision procedure as does the noise. Depending on the symbol sequence ISI will increase or decrease the desired symbol sample.**

**The error probability conditioned on a given intersymbol interference symbol pattern $ISI_i$ is**

$$P\left(e \middle| ISI_i\right) = Q\left( z_k + \sum_{\substack{n=1 \\ n\neq 1}}^{N} c_{ni} z_{k-n} \Big/ \sigma_{n_{rx}} \right) \qquad (81)$$

**The average error probability is**

$$P(e) = \sum_i P\left(ISI_i\right) P\left(e \middle| ISI_e\right) = \frac{1}{M^L} \sum_i P\left(e \middle| ISI_e\right) \qquad (82)$$

**The latter expression is true for equiprobable symbols. M is the number of symbols and L is the length of ISI in symbols.**

# PRACTICAL CALCULATION OF ERROR PROBABILITY WHEN ISI IS PRESENT

**Already with modest values of *M* and *L* the use of the exact error probability formula leads to extensive calculations. During the times a large variety of upper bound methods and approximative methods has been developed. Here only a few examples are given.**

1. **Truncation method: Only a reasonable number of the largest ISI-terms are considered (appoximative method).**
2. **Worst case method: Only the term with the highest ISI-value (sum of the absolute-valued ISI-terms) is considered (upper bound).**

$$P(e) < Q\left(\left(z_k - ISI_{\max}\right)/\sigma_{n_{rx}}\right)$$

3. **Gaussian ISI-model: ISI is considered as Gaussian and the variance of noise and ISI are added giving only one Q-function-term (approximative method).**

$$P(e) \approx Q\left(z_k \Big/ \sqrt{\sigma_{n_{rx}}^2 + \sigma_{ISI}^2}\right)$$

4. **Chernov upper bound.**
5. **Gauss quadrature rule. This method utilizes higher-order moments of ISI (>2) (approximative method)**

# ELIMINATION OF INTERSYMBOL INTERFERENCE IN SYMBOL SEQUENCE SYSTEMS

**ISI can be avoided if the used pulse waveforms are limited to one symbol interval. This requires an ideal channel with infinite bandwidth. Such channels do not exist in the real world, all channels are more or less bandlimited and in addition the may exhibit severe inband amplitude and delay distortions.**
**Signal theory tells us that a signal cannot be simultaneously time- and frequency-limited. In a digital transmission system, however, the pulse need not to be time-limited to one symbol interval, it suffices that the decision circuit input signal disappears on other sampling instants than on the instant when the desired sample is taken.**
**It appears to be possible, at least in theory, to design band-limited pulse waveforms fulfulling this condition. We shall now derive a family of pulses having no ISI, and to present a certain sub-family of pulses which are design goals in many digital transmission systems.**

# INTERSYMBOL INTERFERENCE ELIMINATION

**The the desired pulse signal should fulfil the following requirement:**

$$z(t_s + kT) = \begin{cases} a, & k = 0 \\ 0, & k \neq 0 \end{cases} \quad \textbf{(83)}$$

**It can be shown that a pulse family fulfilling the so called Nyquist 1. criterion**

$$\sum_{l=-\infty}^{\infty} Z\left(f + l/T\right) = \text{constant} = \text{c} \quad \textbf{(84)}$$

**will satisfy the requirement. This can be seen in the following way. By taking the inverse Fourier-transform of both sides of the aliased spectrum above, one obtains**

$$\sum_{l=-\infty}^{\infty} z(t) e^{-j2\pi lt/T} = z(t) \sum_{l=-\infty}^{\infty} e^{-j2\pi lt/T} = c\delta(t) \quad \textbf{(85)}$$

**On the other hand the Fourier-series of an sequence of equidistant impulses is**

$$T \sum_{m=-\infty}^{\infty} \delta(t - mT) = \sum_{l=-\infty}^{\infty} e^{j2\pi lt/T} = \sum_{l=-\infty}^{\infty} e^{-j2\pi lt/T} \qquad \mathbf{(86)}$$

**so the inverse Fourier-transform expression can be written as**

$$z(t)T \sum_{l=-\infty}^{\infty} \delta(t - lT) = c\delta(t) \qquad \mathbf{(87)}$$

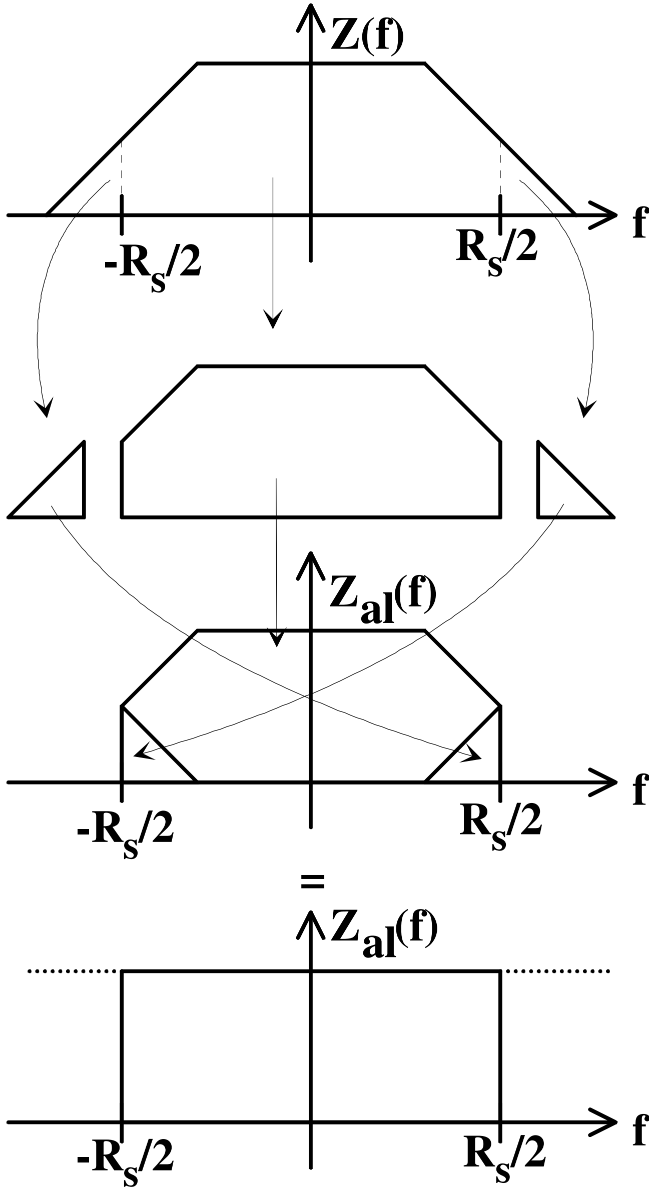
**This equation can be fulfilled only if *z(t)* (after a suitable time shift) has the desired properties giving only one sample differing from zero with equidistant sampling.**

**A signal whose aliased spectrum is constant will be ISI-free if properly sampled.**

# RAISED COSINE SPECTRUM SIGNALING

**Several pulse families having the desired properties may be defined. For instance a pulse having the trapezoidal spectrum shown in the figure below will be ISI-free.**

**The important characteristic is the conjugate mirror symmetry around the Nyquist frequency $R_S/2$.**
**Then the minimum bandwidth giving ISI-free transmission is $R_S/2$. The total response of the system is then an ideal low-pass filter with this bandwidth.**
**The trapezoidal spectral form is not suitable for implementation because it is piecewise linear and its derivative has discontinuity points. One should use pulses changing gradually for easier implementation. In most cases one tries to approximate the raised cosine spectrum whose transition parts are a half cosine wave.**
**No pulse fulfilling the requirements can be exactly implemented because ideal frequency limiting cause the pulses to start at the time $t = -\infty$.**



**Frequency domain expression:**

$$Z(f)=\begin{cases} T, & |f|<(1-\alpha)\dfrac{1}{2T} \\ \dfrac{T}{2}\left[1+\cos\left(\dfrac{\pi}{\alpha}\left(|f|T-\dfrac{1}{2}(1-\alpha)\right)\right)\right], & (1-\alpha)\dfrac{1}{2T}<|f|<(1+\alpha)\dfrac{1}{2T} \\ 0, & |f|>(1+\alpha)\dfrac{1}{2T} \end{cases}$$

**or**

$$Z(f)=\begin{cases} T, & |f|<(1-\alpha)\dfrac{1}{2T} \\ T\left[\cos^2\left(\dfrac{\pi}{2\alpha}\left(|f|T-\dfrac{1}{2}(1-\alpha)\right)\right)\right], & (1-\alpha)\dfrac{1}{2T}<|f|<(1+\alpha)\dfrac{1}{2T} \\ 0, & |f|>(1+\alpha)\dfrac{1}{2T} \end{cases}$$

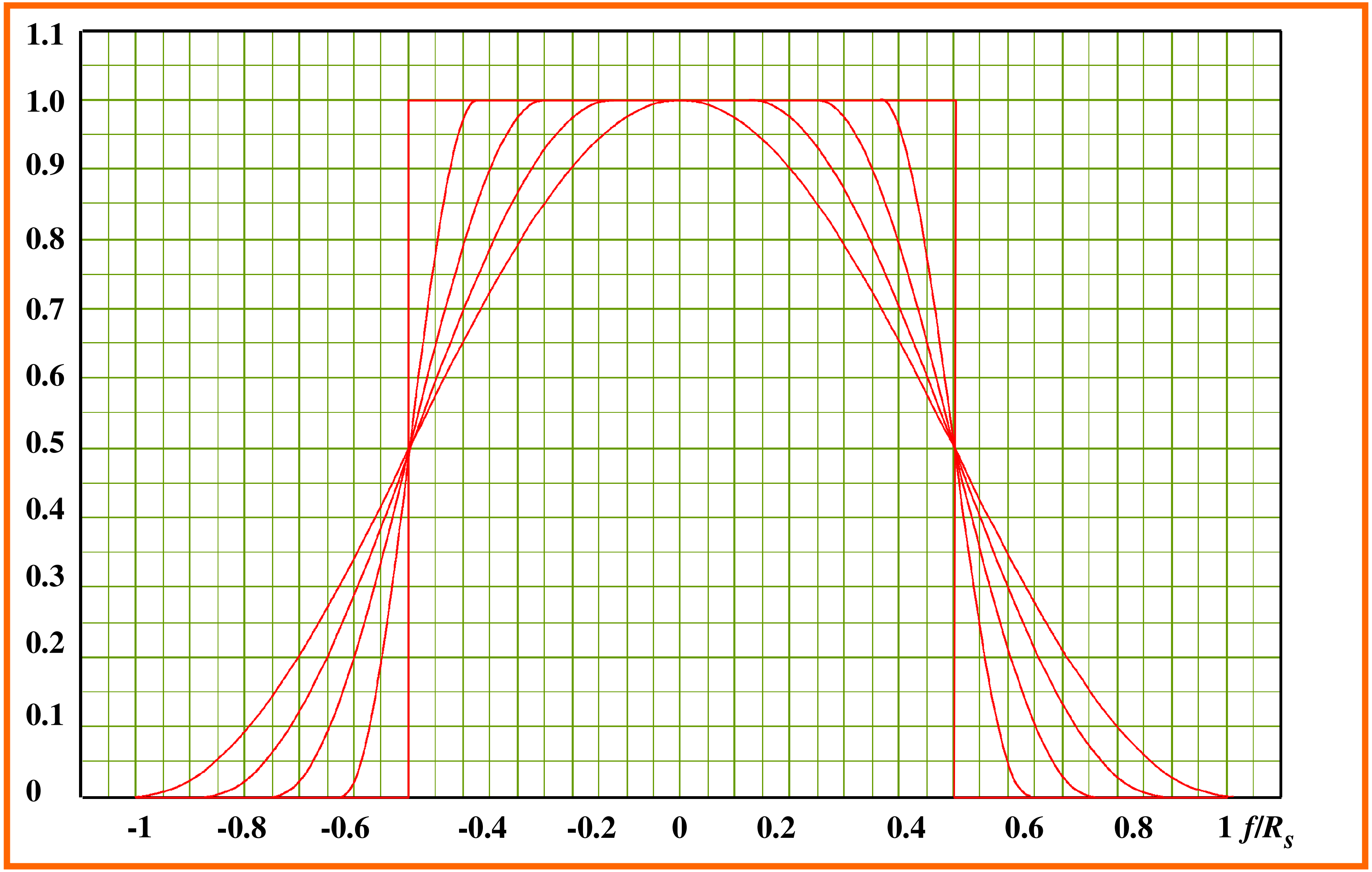
**Time domain expression:**

$$z(t) = \mathrm{sinc}\left(\frac{t}{T}\right)\frac{\cos\left(\pi\alpha\frac{t}{T}\right)}{1-\left(2\alpha\frac{t}{T}\right)^2}$$

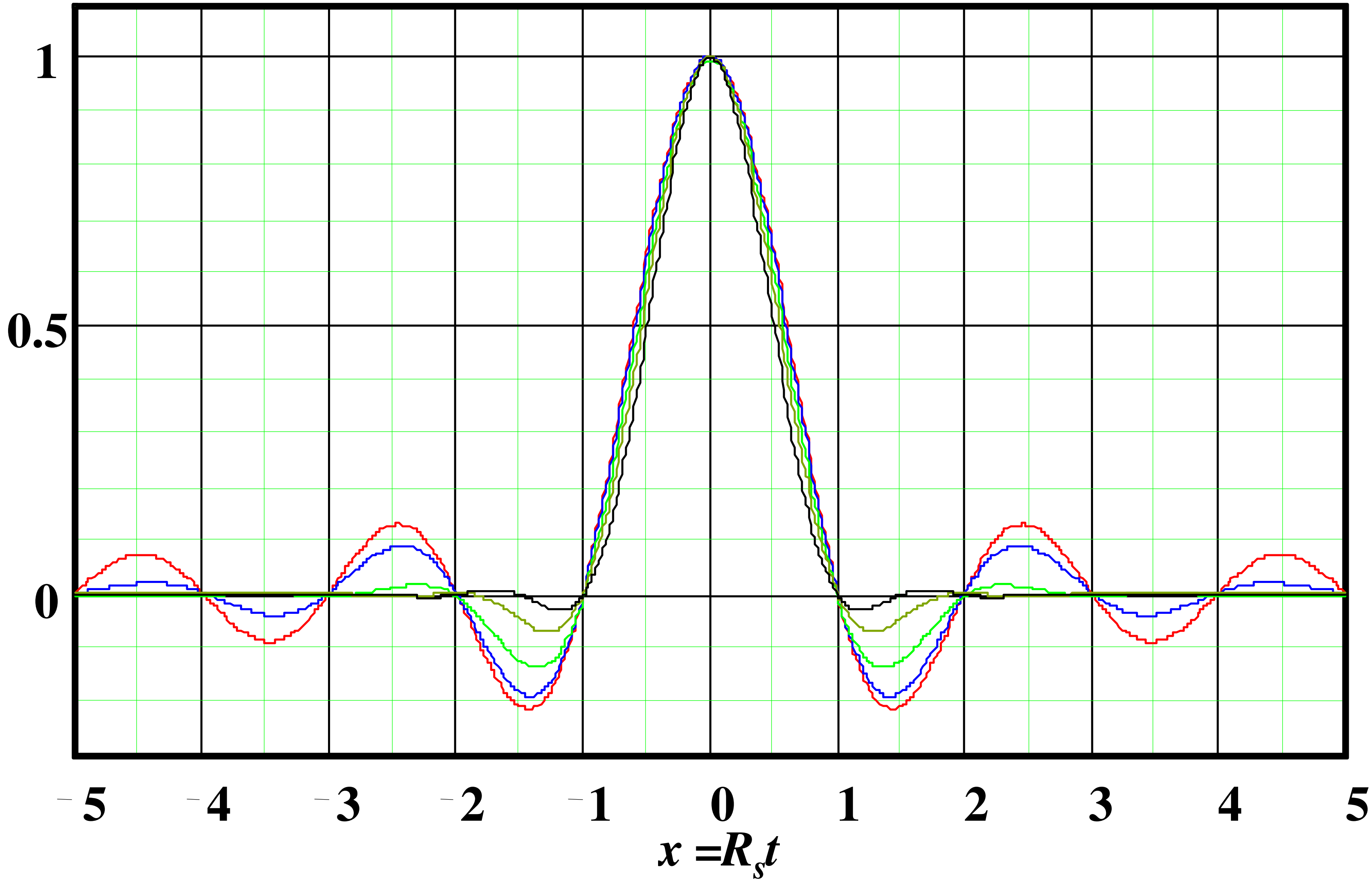
**a is the roll-off parameter giving the total bandwidth** $(1+\alpha)\frac{R_s}{2}$**.**

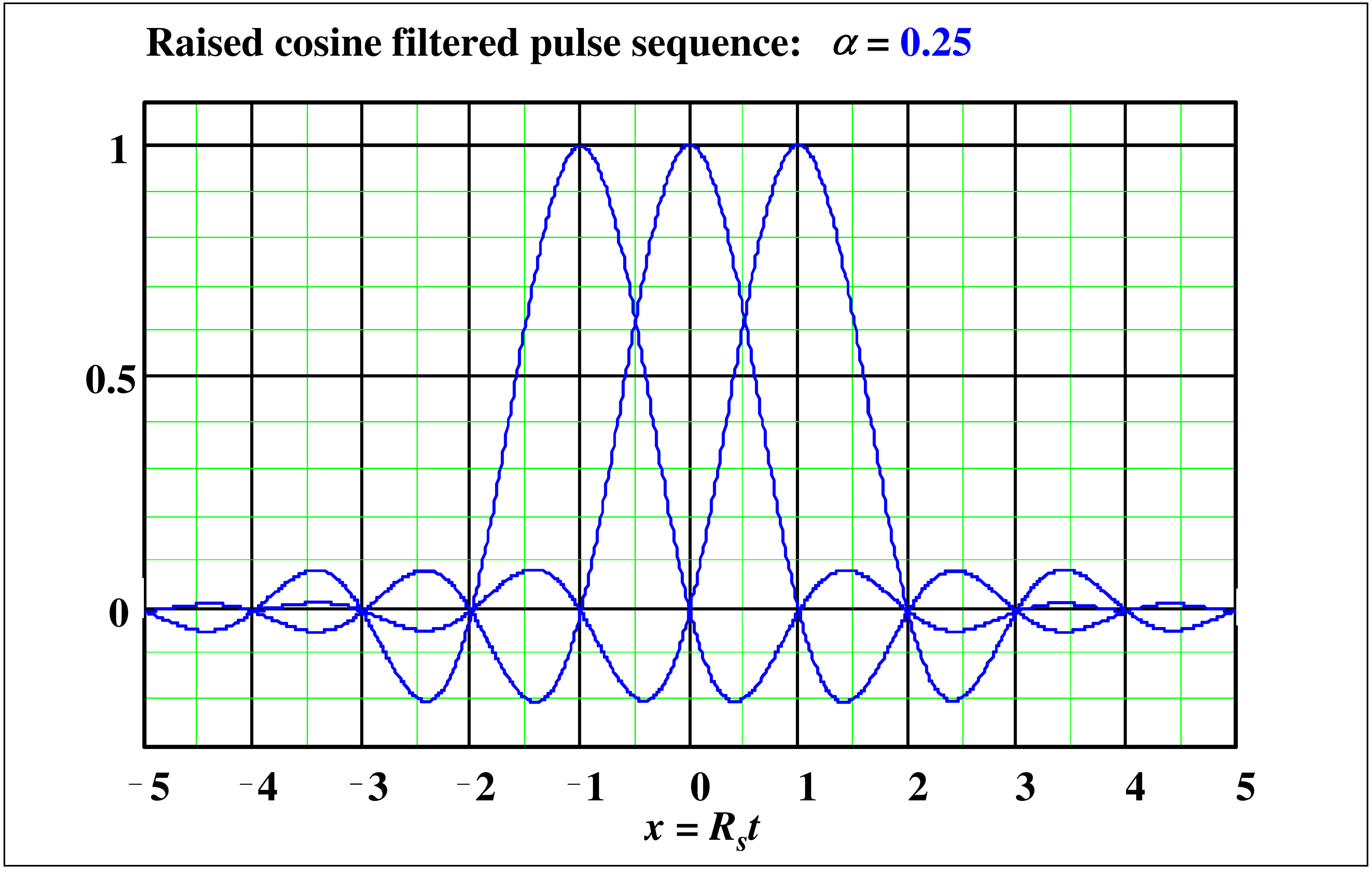
**The performance of matched raised cosine spectrum filtering equals that of the single symbol optimum receiver.**

# Frequency response of raised cosine signals

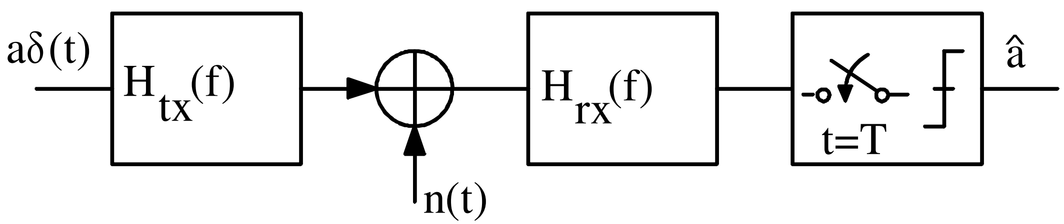


## Raised cosine filter impulse responses: $\alpha = 0, 0.25, 0.5, 0.75, 1$




Raised cosine filtered pulse sequence: $\alpha = 0.25$
1
0.5
0
−5
−4
−3
−2
−1
0
1
2
3
4
5
$x = R_s t$

# SINGLE SYMBOL REFERENCE RECEIVER IN THE ACGN-CHANNEL (Additive Colored Gaussian Noise)




**Signal to noise ratio at the sampling instant is:**

$$\gamma = \frac{s^2}{\sigma^2} = \frac{\left(\int H_{tx}(f) H_{rx}(f) e^{j2\pi fT} df\right)^2}{\int H_{rx}(f) H_{rx}^*(f) S_n(f) df} \qquad \textbf{(91)}$$

**This can be maximized by using the Schwartz inequality:**

$$\int f(x) f^*(x) dx \cdot \int g(x) g^*(x) dx \geq \left| \int f(x) g^*(x) dx \right|^2 \qquad \textbf{(92)}$$

**where γ is maximized when the equality is valid, and this is fulfilled when:**

$$g(x) = c \cdot f^*(x) \qquad \textbf{(93)}$$

**To apply the Schwartz inequality the signal to noise ratio is rewritten as:**

$$\gamma = \frac{\left(\int H_{tx}(f) S_n^{-0,5}(f) H_{rx}(f) S_n^{0,5}(f) e^{j2\pi fT} df\right)^2 \int \left|H_{tx}(f)\right|^2 S_n^{-1}(f) df}{\int \left|H_{rx}(f)\right|^2 S_n(f) df \int \left|H_{tx}(f)\right|^2 S_n^{-1}(f) df} \qquad \textbf{(94)}$$

**This is maximized when** $H_{rx}(f) = \dfrac{H_{tx}^{*}(f) e^{-j2\pi fT}}{S_n(f)}$ **(95)**

**giving the maximum signal to noise ratio** $\gamma = \int \dfrac{\left|H_{tx}(f)\right|^2}{S_n(f)} df$ **(96)**

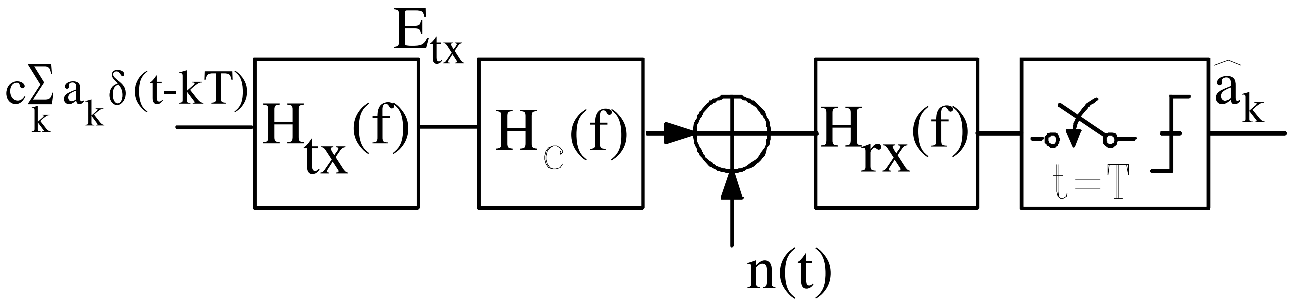
**The transfer function of the receiver filter can be expressed as:**

$$H_{rx}(f) = \frac{1}{\sqrt{S_n(f)}} \cdot \frac{H_{tx}^{*}(f) e^{-j2\pi fT}}{\sqrt{S_n(f)}} \qquad \textbf{(97)}$$

**or the optimum filter consists of a noise whitening filter and a filter mathced to the cascaded transmit filter and whitening filter.**

# SYSTEM OPTIMIZATION ON THE NON-IDEAL TRANSMISSION CHANNEL

**System model**




**The energy of the pulse fed to the transmission channel is:**

$$E_{tx} = c^2 \int \left| H_{tx}(f) \right|^2 df \Rightarrow c = \sqrt{\frac{E_{tx}}{\int \left| H_{tx}(f) \right|^2 df}} \tag{98}$$

**The decision signal sample is (all transfer functions dimensionless):**

$$x(t_s) = \frac{c}{T} \mathbf{F}^{-1}_{t=t_s} \left\{ H_{tx}(f) H_c(f) H_{rx}(f) \right\} \tag{99}$$

**and the variance of the noise sample is:**

$$\sigma^2 = \int \left| H_{rx}(f) \right|^2 S_n(f) df \qquad (100)$$

**When decision is made symbol by symbol one can try in system optimization to eliminate intersymbol interference or minimize the square error either with a given transmit filter or by jointly optimizing the transmit and receive filter with a transmit energy constraint**

# APPROACH I

**Intersymbol interference elimination using raised cosine filtering with given transmit filter.**

**Assume the decision instant to be $t_s = 0$. Using dimensionless filter transfer functions the decision signal sample is:**

$$x(0) = c/T \tag{101}$$

**and the transfer function of the receive filter is**

$$H_{rx}(f) = \frac{X(f)}{H_{tx}(f)H_c(f)} \tag{102}$$

**$X(f)$ is normalized to give $X(0) = 1$.**

**The noise sample variance is**

$$\sigma^2 = \int \left|H_{rx}(f)\right|^2 S_n(f)df = N_o \int_0^{\infty} \frac{\left|X(f)\right|^2}{\left|H_{tx}(f)\right|^2 \left|H_c(f)\right|^2} df \qquad (103)$$

**The second expression is valid in the AWGN-channel.**

**At the decision instant the signal to noise ratio is**

$$\gamma = \frac{c^2 / T^2}{\sigma^2} = \frac{E_{tx} \Big/ \int_{-\infty}^{\infty} \left|H_{tx}(f)\right|^2 df}{T^2 \cdot \frac{N_o}{2} \cdot \int_{-\infty}^{\infty} \frac{\left|X(f)\right|^2}{\left|H_{rx}(f)\right|^2 \left|H_c(f)\right|^2} df}$$

$$= \frac{2E_{tx}}{N_o} \cdot \frac{1}{T^2 \int_{-\infty}^{\infty} \left|H_{tx}(f)\right|^2 df \cdot \int_{-\infty}^{\infty} \frac{\left|X(f)\right|^2}{\left|H_{rx}(f)\right|^2 \left|H_c(f)\right|^2} df} \qquad (104)$$

# APPROACH II

**Elimination of intersymbol interference using raised cosine signalling when the transmit and receive filters are optimized with an average transmit energy constraint.**
**Let the decision instant be $t_s = 0$. Using dimensionless filter transfer functions the decision signal sample is:**

$$x(0) = c/T \tag{105}$$

**The noise power is:** $$\sigma^2 = \int |H_{rx}(f)|^2 S_n(f)df \tag{106}$$

**Transmit energy constraint:** $$E_{tx} = c^2 \int |H_{tx}(f)|^2 df \tag{107}$$

**Transmit filter transfer function is:** $$H_{tx}(f) = \frac{X(f)}{H_{rx}(f)H_c(f)} \tag{108}$$

**which gives** $$E_{tx} = c^2 \int_{-\infty}^{\infty} \frac{|X(f)|^2}{|H_{rx}(f)|^2 \, |H_c(f)|^2} df \qquad (109)$$

**Signal to noise ratio maximization can be performed with variational techniques.**
**First, the receive filter is optimized. To simplify the expressions following functions are defined:**

$$\begin{aligned} S(f) &= |H_{tx}(f)|^2 \\ C(f) &= |H_c(f)|^2 \\ R(f) &= |H_{rx}(f)|^2 \\ Q(f) &= |X(f)|^2 \end{aligned} \qquad (110a,b,c,d)$$

**The optimun transfer function and the additional variation is:**

$$R(f) = R_o(f) + \alpha \cdot \Delta R(f) \qquad (111)$$

**In the variational calculus the minimizing function added with the constraint weigthed by a Lagrange-multiplier is differentiated:**

$$\frac{d}{d\alpha}\int_{-\infty}^{\infty}\left\{\left(R_o(f)+\alpha\cdot\Delta R(f)\right)S_n(f)+\lambda c^2\frac{Q(f)}{\left[R_o(f)+\alpha\cdot\Delta R(f)\right]C(f)}\right\}$$
$$=\int_{-\infty}^{\infty}\left\{\Delta R(f)S_n(f)-\lambda c^2\frac{Q(f)}{C(f)}\frac{\Delta R(f)}{\left[R_o(f)+\alpha\cdot\Delta R(f)\right]^2}\right\}df \qquad \textbf{(112)}$$

**Putting $\alpha = 0$ one gets in the zero of the derivative:**

$$\int_{-\infty}^{\infty}\left\{\Delta R(f)\left[S_n(f)-\lambda c^2\frac{Q(f)}{C(f)R_o(f)}\right]\right\}df = 0 \qquad \textbf{(113)}$$

**Now the power transfer function of the optimum receive filter can be solved:**

$$R_o(f)=\sqrt{\lambda c^2\frac{Q(f)}{C(f)S_n(f)}} \Rightarrow S_o(f)=\frac{Q(f)}{R_o(f)C(f)}=\sqrt{\frac{Q(f)S_n(f)}{\lambda c^2 C(f)}} \qquad \textbf{(114a,b)}$$

**Finally the noise power is calculated:**

$$\sigma^2 = \int \left| H_{rx}(f) \right|^2 S_n(f) df$$

$$= \int_{-\infty}^{\infty} \sqrt{\lambda c^2 \frac{Q(f)}{C(f) S_n(f)}} \cdot S_n(f) df = \int_{-\infty}^{\infty} \sqrt{\lambda c^2 \frac{Q(f) S_n(f)}{C(f)}} df \quad \textbf{(115)}$$

**The parameter $\lambda c^2$ can be calculated from the energy constraint:**

$$E_{tx} = c^2 \int \left| H_{tx}(f) \right|^2 df = c^2 \int_{-\infty}^{\infty} \sqrt{\frac{Q(f) S_n(f)}{\lambda c^2 C(f)}} df \quad \textbf{(116)}$$

$$\sqrt{\lambda c^2} = \frac{c^2}{E_{tx}} \int_{-\infty}^{\infty} \sqrt{\frac{Q(f) S_n(f)}{C(f)}} df \quad \textbf{(117)}$$

**Inserting these expressions in the expression for the signal to noise ratio one gets:**

$$\sigma^2 = \int_{-\infty}^{\infty} \frac{c^2}{E_{tx}} \int_{-\infty}^{\infty} \sqrt{\frac{Q(\nu)S_n(\nu)}{C(\nu)}} d\nu \cdot \sqrt{\frac{Q(f)S_n(f)}{C(f)}} df$$

$$= \frac{c^2}{E_{tx}} \left( \int_{-\infty}^{\infty} \sqrt{\frac{Q(f)S_n(f)}{C(f)}} df \right)^2 \tag{118}$$

**The expression for the signal to noise ratio is now:**

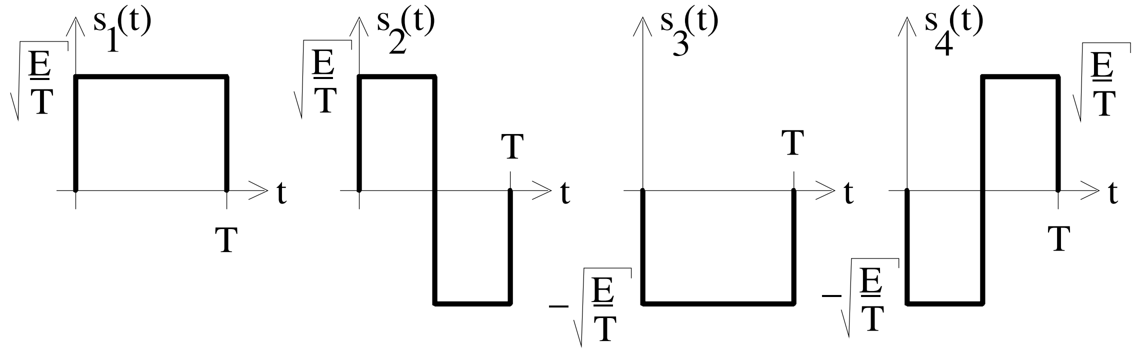
$$\gamma = \frac{\frac{c^2}{T^2}}{\frac{c^2}{E_{tx}} \left( \int_{-\infty}^{\infty} \sqrt{\frac{Q(f)S_n(f)}{C(f)}} df \right)^2} = \frac{E_{tx}}{\left( T \int_{-\infty}^{\infty} \sqrt{\frac{Q(f)S_n(f)}{C(f)}} df \right)^2} \tag{119}$$

**In the AWGN-channel the signal to noise ratio expression is reduced to**

$$\gamma = \frac{2E_{tx}}{N_o} \frac{1}{\left( T \int_{-\infty}^{\infty} \frac{|X(f)|}{|H_c(f)|} df \right)^2} \tag{120}$$

Homeworks

1 In a four-symbol digital system with equally probable symbols the pulses in the figure are used in transmission over a AWGN-channel.




a) Convert the pulse waveforms into signal vectors using rectangular pulses as basis functions.

b) Sketch the optimum receiver using matched filters. Draw the filter impulse responses.

c) Using the vector representation draw the signal constellation in the $r_1$ -$r_2$-coordinate system and derive the SEP-expression.

d) Derive the exact BEP-expression when Gray-coding is used

2 In baseband binary transmission the symbol values -1 and +1 are equiprobable. The decision circuit input noise is Laplace-distributed (symmetrically exponentially distributed):

$$p(n) = \frac{1}{\sigma\sqrt{2}} \exp\left(\frac{|n|\sqrt{2}}{\sigma}\right)$$

where $\sigma$ is the r.m.s. noise amplitude.

a) Derive the BEP-expression when the signal samples are ±s.

b) Calculate how many dB better the signal to noise ratio must be in the Laplace-channel than in the Gaussian channel on the BEP-levels $10^{-3}$, $10^{-6}$ and $10^{-9}$.

Note! $Q(3.09) = 10^{-3}$, $Q(4.75) = 10^{-6}$, $Q(6.00) = 10^{-9}$

3 A raised-cosine spectrum pulse has a half amplitude bandwidth of 1200 Hz and the spectrum upper limit is 1500 Hz. The pulse is used for transmission of 4 state digital signal.

a) Determine the roll-off factor.

b) What is the bit rate of the transmitted signal?

The signal is transmitted over a multiple echo channel with the impulse response

$$h(t) = -0.2\delta(t+T) + \delta(t) + 0.4\delta(t-T) - 0.3\delta(t-2T)$$

where $T$ is the inverse of the symbol rate.

c) Which is the symbol sequence causing the maximum intersymbol interference (ISI), and what is the magnitude of the maximum ISI? Sampling is assumed to take place at same time instant as in the ideal channel.

d) What is the occurrence probability of the symbol sequence causing the maximum ISI?

4 In a bipolar binary system rectangular pulse (duration T) sequences are transmitted through an AWGN-channel with the pulse rate 1/T, and the receiver uses a matched filter. There is a timing error ΔT in decision sampling causing ISI. How large may ΔT/T be, that the degradation should not exceed 1.0 dB on the BEP-level $10^{-3}$?