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LTV System Modelling

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Contents

1. Introduction

Linear elements in a communication system can be time-invariant (LTI) or time varying (LTV) in nature. The assumption of time invariance implies that the properties of systems being modelled do not change over (long periods of) time.

Whether to use a time-invariant or time-varying model is usually determined by the rate at which the characteristics of the communications system being modelled are changing in comparison to other parameters of the communications system such as the symbol rate. If the system parameters are changing at a rate approaching the symbol rate, a time-varying model is appropriate.

The fixed radio link channel characteristics are due to changes in the atmospheric conditions, which typically have a time constant of several minutes to hours. If the communication link is operating at a symbol rate of 100 Mbit/s, then the time constant associated with the channel variations is very long compared to the symbol time. If the objective of simulation is BER estimation, then, during the estimation interval, the channel can be assumed to be in a static state and a time-invariant model can be used.

The long-term behaviour of the channel and its impact on long-term system performance can be evaluated by analysing system performance over a series of snapshots of static channel conditions, using a different time-invariant model for each snapshot.

In the other hand, when the radio link channel is affected by multipath phenomena, then very fast changes in the channel characteristics may occur and time-varying model must be used. While a time-varying model may not be needed for BER estimation, such a model will be necessary to study the behaviour of receiver subsystems such as synchronisers and equalisers. Also, in the mobile radio channel the movement of the mobile terminal causes fast changes in the characteristics of the channel compared to the transferred symbol rate.

2. Time-Domain Description for Linear Time-Varying Systems

2.1. The Impulse Response

The commonly used form of the time-varying impulse response is modelled on the definition of the impulse response for linear time-invariant systems as

$$h(t, t\ ) = \Gamma\left[d\,(t - t\ )\right] \qquad (1)$$

where t is the time (from the origin) when the impulse is applied, and Γ is the linear time-varying system operator.

The most important application of time varying models in communication systems is in the modelling of time-varying (multipath) channels. To describe the behaviour of such channels, Kailath introduced several alternative formulations for the impulse response. The most convenient for purposes of modelling communication channels is to define $c(\hat{t}, t)$ as the response measured at time t to a unit impulse applied at time $t - \hat{t}$,

$$c(\hat{t}, t) = \Gamma\left[d\,(t - (t - \hat{t}\,))\right] = \Gamma\left[d\,(\hat{t}\,)\right] \qquad (2)$$

The above impulse response is referred to as the *channel impulse response*.

The relationship of the definition of the impulse response to the channel impulse response is $c(\hat{t}, t) = h(t, t - t\ )$ or $h(t, t\ ) = c(t - \hat{t}, t)$.

Since the system is time-varying, the impulse response will change as a function of both the time at which the impulse is applied and the time at which the output is measured. Hence both $c(\hat{t},t)$ and $h(t,t)$ are surfaces in three-dimensional space. Cross sections of these surfaces are illustrated in Figure 1.

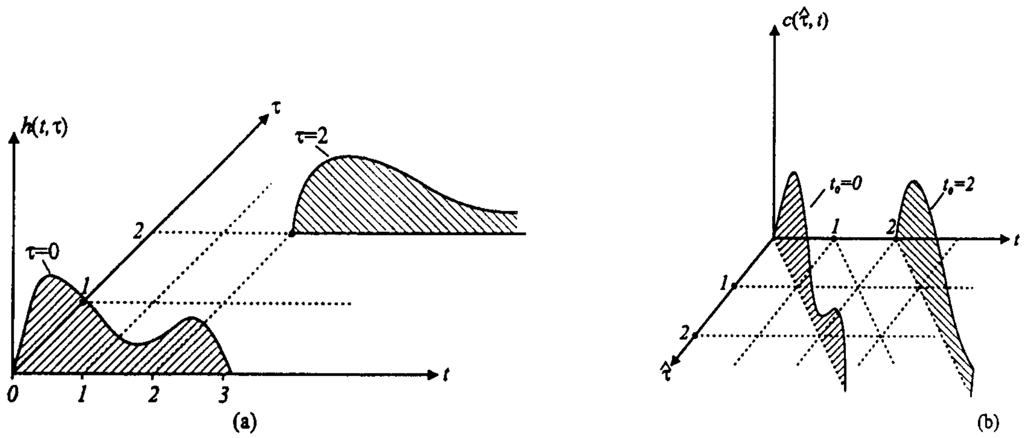

Figure 1: Two-dimensional representation of a time-varying impulse response

Figure 1a shows two hypothetical impulse responses $h(t,t)$ for values of t = 0, 2. Observe that causality requires $h(t,t)$ = 0 for $t < t$. Because of the relationships $c(\hat{t},t)$ = $h(t,t-t)$ and $\hat{t} = t - t$, it is simple to establish a correspondence between the two functions.

While the impulse response of an LTI system maintains the same functional form irrespective of when the impulse was applied at the input, the impulse response of an LTV system depends on when the input was applied.

2.2. The Superposition Integral

The response of the system to an arbitrary input is determined by a superposition integral. Using the impulse response, the superposition integral is

$$y(t) = \int_{-\infty}^{\infty} h(t,t)x(t)dt \qquad (3)$$

or

$$y(t) = h(t,t) * x(t) \qquad (4)$$

In terms of the channel response we have

$$y(t) = \int_{-\infty}^{\infty} c(\hat{t}, t) x(t - \hat{t}) d\hat{t} \qquad (5)$$

or

$$y(t) = c(\hat{t}, t) * x(t - \hat{t}) \qquad (6)$$

When the system is time-invariant, $h(t,t) = h(t-t)$ and $c(\hat{t},t) = c(\hat{t})$, and (3),(5) is the convolution integral. Although the convolution integral is also sometimes referred to as the superposition integral, we will reserve the term *superposition* for the time-varying case in (3) and (5). For causal systems this integral may be written as

$$y(t) = \int_{0}^{\infty} h(t,t) x(t) dt \quad because \quad h(t,t) = 0 \quad for \quad t < 0 \qquad (7)$$

and

$$y(t) = \int_{0}^{\infty} c(\hat{t}, t) x(t - \hat{t}) d\hat{t} \quad where \quad c(\hat{t}, t) = 0 \quad for \quad \hat{t} > t \qquad (8)$$

3. Frequency-Domain Representations of Time-Varying Systems

For an LTV system one can develop the notion of a transfer function, though a time-varying transfer function $C_{\hat{t}}(f,t)$, by simply taking the Fourier transform of $c(\hat{t},t)$ with respect to $\hat{t}$ as

$$C_{\hat{t}}(f, t) = \int_{-\infty}^{\infty} c(\hat{t}, t) \quad e^{-j2pf\hat{t}} d\hat{t} \qquad (9)$$

If the system is slowly time-varying, then the concepts of frequency response and bandwidth can be applied to $C_{\hat{t}}(f,t)$. Whereas the LTI system is characterised by a single impulse response function and a single transfer function, the LTV system is characterised by a family of impulse response functions and transfer functions, one function for each value of *t*, as shown in Figure 1.

The inverse transform is given by

$$c(\hat{t},t) = \int_{-\infty}^{\infty} C_{\hat{t}}(f,t) e^{j2pf\hat{t}} df \qquad (10)$$

The output *y(t)* of a system $c(\hat{t},t)$ with the input *x(t)* can be determined by

$$y(t) = \int_{-\infty}^{\infty} C_{\hat{t}}(f,t) X(f) e^{j2pft} df \qquad (11)$$

where *X(f)* is the Fourier transform of *x(t)*.

3.1. Two-Dimensional Frequency response

Kailath introduced a frequency-domain function

$$C_{\hat{t},t}(f,n) = \int_{-\infty}^{\infty} C_{\hat{t}}(f,t)\; e^{-j2pnt} dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(\hat{t},t)\; e^{-j2pf\hat{t}} e^{-j2pnt} dt \quad dt \qquad (12)$$

The frequency n is associated with the rate of change of the system and f is associated with the frequency of the system excitation.

From (10) and (11) we get the system output frequency response

$$Y(n) = \int_{-\infty}^{\infty} C_{\hat{t},t}(f, n-f) X(f) df \qquad (13)$$

Equation (13) is the frequency-domain convolution of the input and system frequency-domain characteristics. In the double integral of Equation (12), the frequency variable f is associated with the time variable $\hat{t}$ and it may be viewed as analogous to the frequency variable f in the transfer function *H(f)* of linear time-invariant systems.

3.2. Bandwidth Relations in Time-Varying Systems

The time-varying character of systems is usually manifested by a frequency spread, or a frequency shift, or both. Thus the response of a LTV system to a single frequency f_0 can be a frequency spectrum or a frequency spectrum centered about a different frequency than f_0. The width of the spectrum can be regarded as a measure of the variation of the system.

3.3. Sampling Rate

An input with bandwidth B_i to a linear time-varying system characterised by B_s results in an output bandwidth not greater than $B_i + B_s$. Hence, the sampling rate must be

$$f_s = \frac{1}{T_s} \geq 2(B_i + B_s) \tag{14}$$

4. Properties of Linear Time-Varying Systems

4.1. Properties of Convolution

The properties of convolution for LTI systems are: associativity, distributivity and commutativity. For LTV systems the properties are associativity and distributivity, the commutativity does not hold:

$$h_1(t,t') * h_2(t,t') \neq h_2(t,t') * h_1(t,t') \tag{15}$$

4.2. Interconnections of Linear Time-Varying Systems

As in the case of LTI systems, the interconnections of LTV systems can be simplified using block diagram equivalents although the simplification is much more complicated since transform methods are not applicable.

A cascade interconnection of systems is equivalent to the superposition of their impulse responses. For LTV systems the cascade operation is not commutative.

5. Models for LTV Systems

5.1. Linear Differential Equation with Time-Varying Coefficients

Some LTV systems are described by linear differential equations with time-varying coefficients of the form

$$a_n(t)\frac{d^n y(t)}{dt^n} + a_{n-1}(t)\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_0(t)y(t) = x(t) \tag{16}$$

If the system is slowly varying it can be regarded as quasistatic. It can then be modelled as a recursive IIR filter.

For rapidly varying systems the impulse response is

$$h(t,t\,) = y(t,t\,)u(t - t\,) \qquad (17)$$

If (16) has an analytical solution, its impulse response has a separable form

$$h(t,t\,) = \sum_{i=1}^{n} p_i(t)q_i(t\,), \quad t \geq t \qquad (18)$$

The above represents a separation model. Unfortunately, only first order differential equations have a general solution.

5.2. Separable Models

If the system has a separable form of impulse response, then the system output is

$$y(t) = \int_0^t h(t,t\,)x(t\,)dt \; = \sum_{i=1}^{N} p_i(t)\int_0^t x(t\,)q_i(t\,)dt \qquad (19)$$

The realisation of such a system is shown in Figure 2.

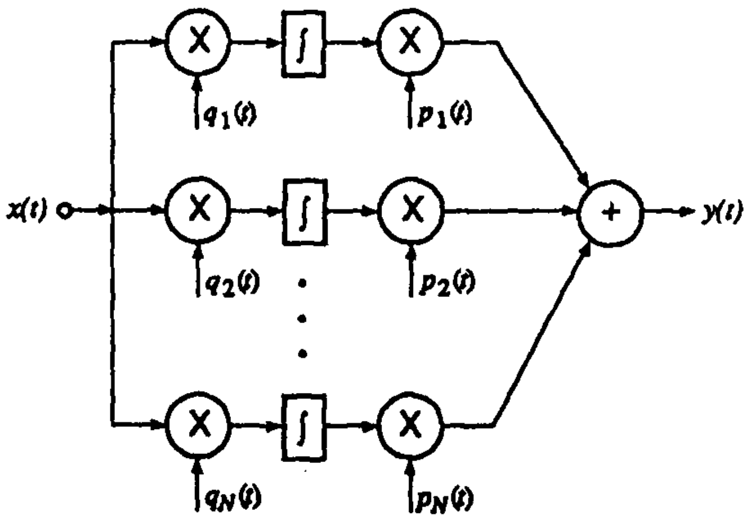

Figure 2: Structure of separable models for LTV systems.

5.3. Tapped Delay-Line Channel Models

A variety of tapped delay-line models have been developed by Kailath based on the sampling theorem. These models differ depending upon whether one assumes the input to the channel, the output or the channel itself to be bandlimited or not. If the input signal is bandlimited, a tapped delay-line can be derived either for lowpass or bandpass channels.

If the channel input x(t) is bandlimited to B_i, the output y(t) is

$$y(t) = \frac{1}{2B_i} \sum_{m=-\infty}^{\infty} g\left[\frac{m}{2B_i}, t\right] x\left[t - \frac{m}{2B_i}\right] \qquad (20)$$

The representation (20) can be synthesised by a tapped delay-line with taps having time-varying gains $g_n(t) = g\left(\frac{n}{2B_i}, t\right)$ and tap delay $T = \frac{1}{2B_i}$, as shown in the Figure 3.

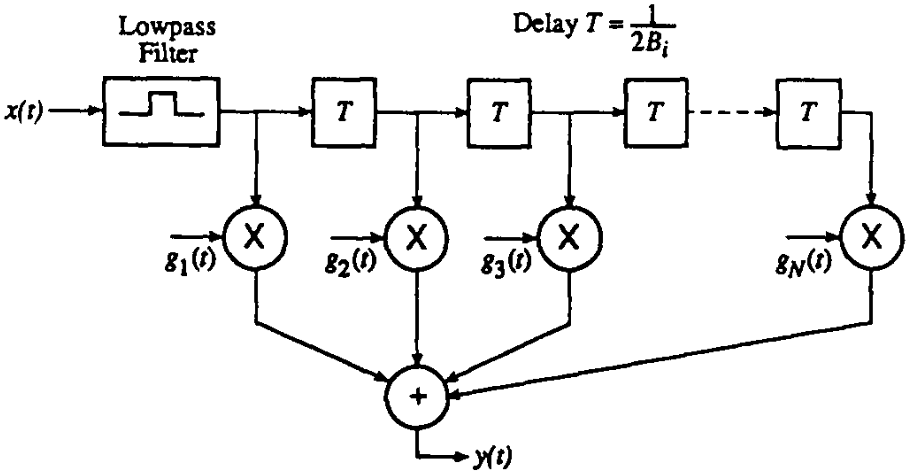

Figure 3: Sampling model for LTV system in the form of tapped delay-line.

In simulation the quasistatic approach is feasible if the channel Doppler spread is much smaller than the signal bandwidth, i.e. $B_s << B_i$. The tap gains can then be regarded as constant and the simulation of fading system can be approximated by a series of simulations.

6. Conclusion

Linear elements in a communication system can be time-invariant (LTI) or time varying (LTV) in nature. Radio channels are usually time varying. Slow variations are caused by changes in the atmospheric conditions and rapid variations occur in multipath propagation events or when the radio terminals are moving in the mobile radio network.

LTV systems are more difficult to simulate than LTI systems. In LTV systems the bandwidth expands, which is called the Doppler broadening, and therefore the simulation sampling rate must be increased accordingly. If the input signal is bandlimited the LTV system can be represented as a tapped delay-line with time-varying coefficients.

References

Michel C. Jeruchim, Philip Balaban & K. Sam Shanmugan: Simulation of Communication Systems. Second Edition. Kluwer Academic / Plenum Publishers. New York. 2000.

Problem

Show that the relationship in $Y(n) = \int_{-\infty}^{\infty} C_{f,t}(f, n - f) X(f) df$ (system output frequency response) implies the one in $f_s = \frac{1}{T_s} \geq 2(B_i + B_s)$ (sampling rate).