Baseband equivalents of bandpass signals

Jukka Henriksson

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- Complex baseband representation of signals
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- Channel models
- Examples of modulated signals
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If time allows

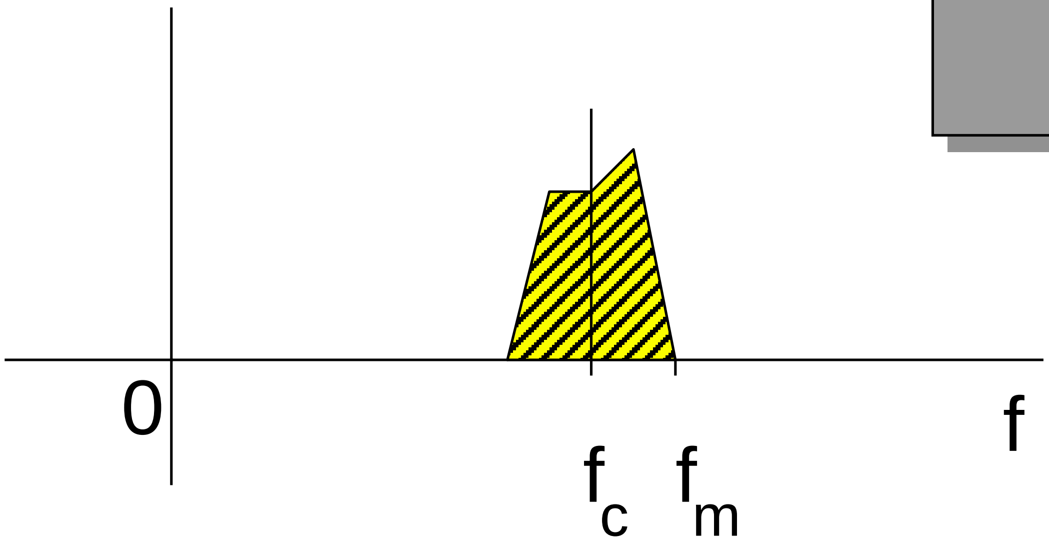
Background, motivation and goals

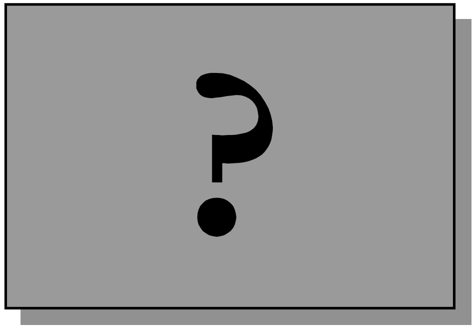
- Simulation requires digital representation of signals
- sampling in time domain
- efficient representation needed
- minimum amount of samples
- noise (and interference) should be correctly included

The way: complex baseband representation

Background, motivation and goals (cont'd)

- Direct application of sampling theorem
- $\Rightarrow$ required sampling frequency $f_s > 2 \cdot f_m$ (??)





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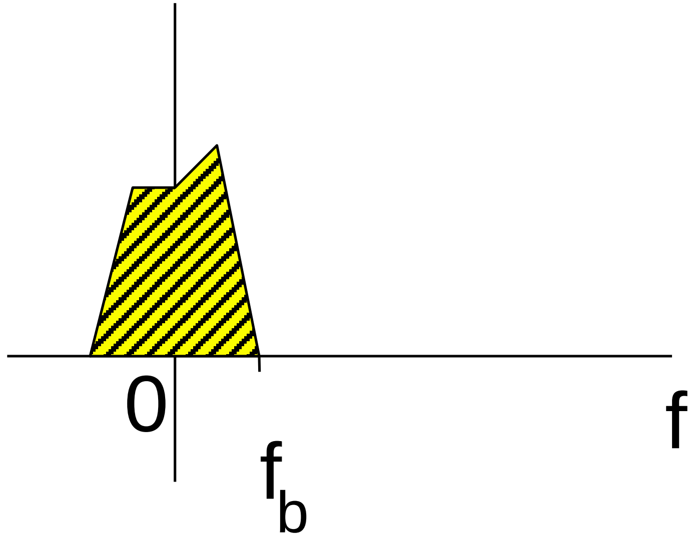
Background, motivation and goals (cont'd)



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- For complex bb-representation only $f_s > 2 \cdot f_b$ is needed

Useful complex relationships

- Define complex number:

$$z = x + jy$$

- Then

$$\mathrm{Re}\{z_1\} \cdot \mathrm{Re}\{z_2\} = \frac{1}{2}\left(z_1 + z_1^*\right) \cdot \frac{1}{2}\left(z_2 + z_2^*\right) \quad (1)$$

$$= \frac{1}{2}\left(\mathrm{Re}\{z_1 z_2\} + \mathrm{Re}\{z_1 z_2^*\}\right)$$

- and

$$\mathrm{Re}\{z_1\} \cdot \mathrm{Im}\{z_2\} = \frac{1}{2}\left(z_1 + z_1^*\right) \cdot \frac{1}{2j}\left(z_2 - z_2^*\right) \quad (2)$$

$$= \frac{1}{2}\left(\mathrm{Im}\{z_1 z_2\} - \mathrm{Im}\{z_1 z_2^*\}\right) = \frac{1}{2}\left(\mathrm{Im}\{z_1 z_2\} + \mathrm{Im}\{z_1^* z_2\}\right)$$

Useful complex relationships (cont'd)

- and

$$\mathrm{Im}\{z_1\}\cdot\mathrm{Im}\{z_2\}=\frac{1}{2j}\left(z_1-z_1^*\right)\cdot\frac{1}{2j}\left(z_2-z_2^*\right)$$

$$=\frac{1}{2}\left(-\mathrm{Re}\{z_1z_2\}+\mathrm{Re}\{z_1z_2^*\}\right)$$

(3)

- and also

(4)

$$\mathrm{Re}\{z_1z_2\}=x_1x_2-y_1y_2=\mathrm{Re}\{z_1\}\mathrm{Re}\{z_2\}-\mathrm{Im}\{z_1\}\mathrm{Im}\{z_2\}$$

(5)

$$\mathrm{Im}\{z_1z_2\}=x_1y_2+y_1x_2=\mathrm{Re}\{z_1\}\mathrm{Im}\{z_2\}+\mathrm{Im}\{z_1\}\mathrm{Re}\{z_2\}$$

Complex representation of bandpass signal

Assume bandpass signal

$$v(t) = A(t)\cos(\omega_c t + \phi(t)) \qquad (6)$$

this is equivalent to

$$\begin{aligned} v(t) &= \mathrm{Re}\left\{A(t)e^{j\omega_c t + j\phi(t)}\right\} = \mathrm{Re}\left\{\tilde{z}(t)e^{j\omega_c t}\right\} \\ &= \tfrac{1}{2}\left[\tilde{z}(t)e^{j\omega_c t} + \tilde{z}(t)^* e^{-j\omega_c t}\right] \end{aligned} \qquad (7)$$

where the ***complex envelope*** is defined

$$\boxed{\tilde{z}(t) = A(t)e^{j\phi(t)}} \qquad (8)$$

Alternative real representation

BP-signal of (7) may be written also

$$v(t) = x(t)\cos(\omega_c t) - y(t)\sin(\omega_c t) \quad (9)$$

where complex envelope has been given using

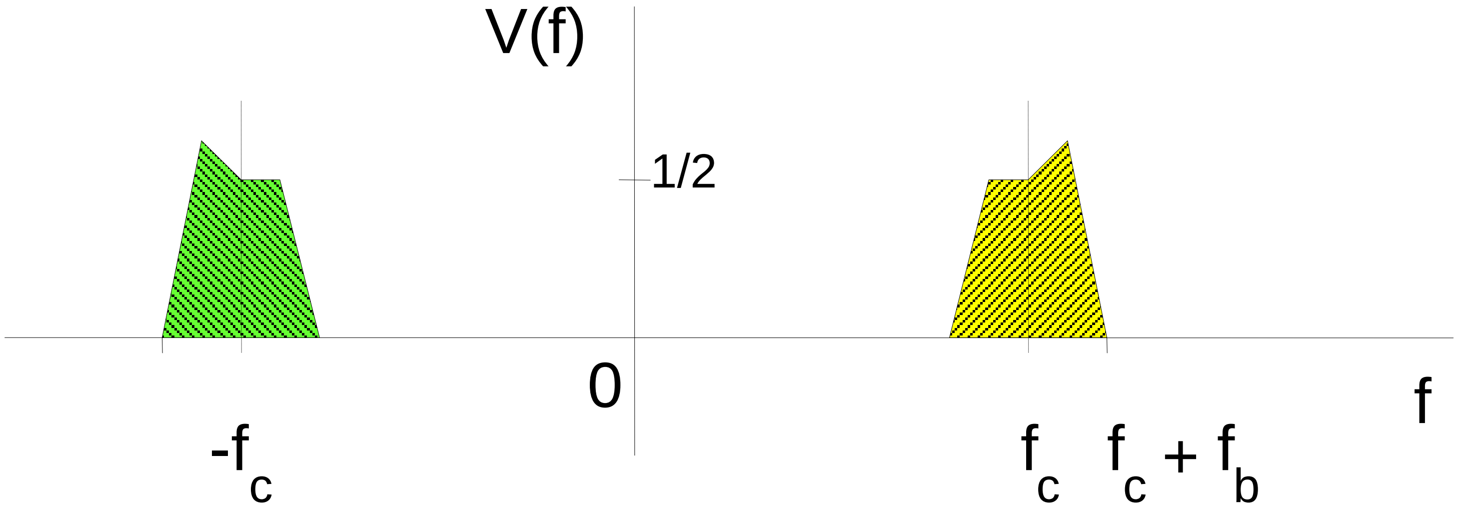
inphase (I) and *quadrature* (Q)

components

$$\tilde{z}(t) = x(t) + jy(t) \quad (10)$$

Spectral considerations

- If v(t) is bandpass => A(t) and ϕ(t) are slowly varying lowpass type
- If ω_c is the center frequency of the bandpass signal and $\omega_c > \omega_b$ => complex envelope is unique



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Spectral considerations (cont'd)

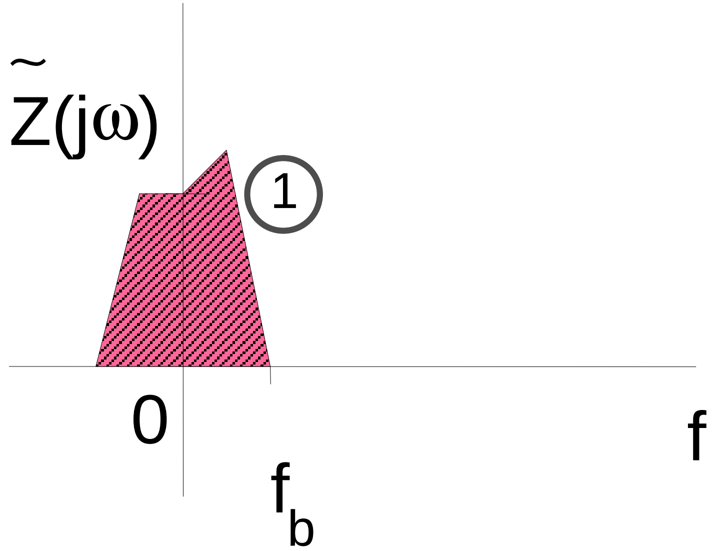
Fourier transform of (7) gives

$$V(f) = \tfrac{1}{2}\left[\tilde{Z}(f - f_c) + \tilde{Z}(-f - f_c)^*\right] \qquad (11)$$

so the bandpass spectrum can be given using only information of $\tilde{Z}(f)$

◁ >15

Spectrum of the complex low-pass signal



Notice the spectrum height!

Recovery of quadrature component x

$$v(t)\,\mathrm{Re}\left\{e^{-j\omega_c t}\right\} = \tfrac{1}{2}\mathrm{Re}\left\{\tilde{z}(t)\right\} + \tfrac{1}{2}\mathrm{Re}\left\{\tilde{z}(t)e^{j2\omega_c t}\right\}$$
$$= \tfrac{1}{2}x(t) + \tfrac{1}{4}\left[\tilde{z}(t)e^{j2\omega_c t} + \tilde{z}(t)^* e^{-j2\omega_c t}\right] \qquad (12)$$

Fourier transforming we get

$$\tfrac{1}{2}X(f) + \tfrac{1}{4}\left[\tilde{Z}(f-2f_c) + \tilde{Z}(-f-2f_c)^*\right] \qquad (13)$$

The transform components are bandlimited ⇨

lowpassfiltering the result we get 1/2 X(f) or

$$x(t) = \left[2v(t)\,\mathrm{Re}\left\{e^{-j\omega_c t}\right\}\right]_{LP} \qquad (14)$$

(11)

Recovery of quadrature component y

$$v(t)\,\mathrm{Im}\left\{e^{-j\omega_c t}\right\} = \tfrac{1}{2}\,\mathrm{Im}\left\{\tilde{z}(t)\right\} - \tfrac{1}{2}\,\mathrm{Im}\left\{\tilde{z}(t)e^{j2\omega_c t}\right\}$$
$$= \tfrac{1}{2}\,y(t) - \tfrac{1}{4j}\left[\tilde{z}(t)e^{j2\omega_c t} - \tilde{z}(t)^* e^{-j2\omega_c t}\right] \qquad (15)$$

Fourier transforming we get

$$\tfrac{1}{2}\,Y(f) + \tfrac{1}{4j}\left[\tilde{Z}(f-2f_c) - \tilde{Z}(-f-2f_c)^*\right] \qquad (16)$$

and hence

$$\boxed{y(t) = \left[2v(t)\,\mathrm{Im}\left\{e^{-j\omega_c t}\right\}\right]_{LP}} \qquad (17)$$

Spectra of low-pass components

Starting from (14) Fourier-transforming, using (11) we get

$$X(f) = \tfrac{1}{2}\left[\tilde{Z}(f) + \tilde{Z}(-f)^*\right] \tag{18}$$

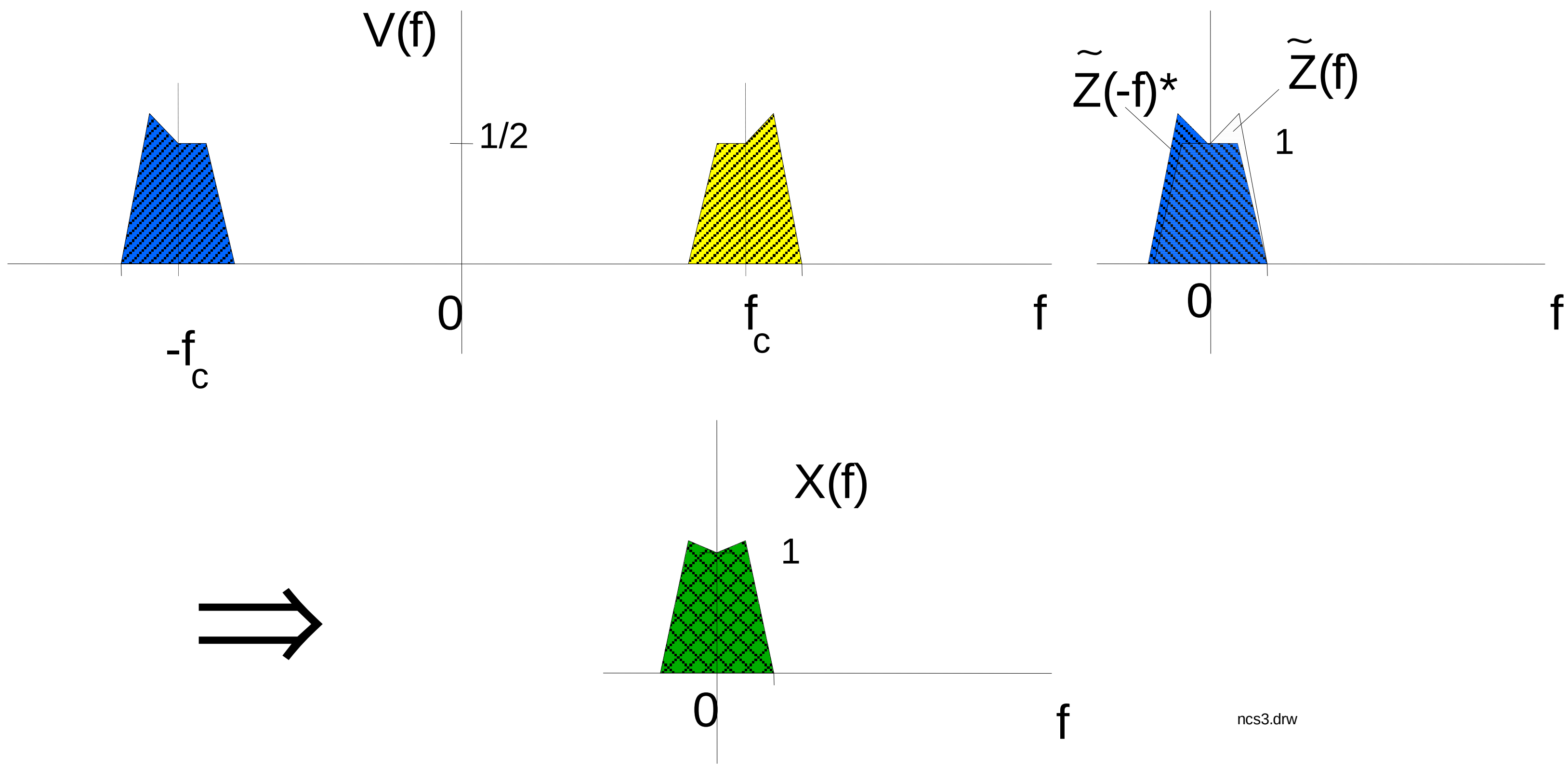
and similarly

$$Y(f) = \frac{1}{2j}\left[\tilde{Z}(f) - \tilde{Z}(-f)^*\right] \tag{19}$$

The following figure demonstrates above.

(Note: x(t) and y(t) real => X(f) and Y(f) conjugate symmetric i.e. X(-f)=X(f)*)

Derivation of X(f) from BP spectrum V(f)



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Linear filtering of BP signals

Assume BP-filter impulse response

$$h(t) = \mathrm{Re}\left\{ 2\tilde{h}(t)e^{j\omega_c t} \right\} \tag{20}$$

Let input be v(t) – then the output w(t) is

$$w(t) = \int h(t-\tau)v(\tau)d\tau \tag{21}$$

$$= \int \tfrac{1}{2}\mathrm{Re}\left\{ 2\tilde{h}(t-\tau)e^{j\omega_c(t-\tau)}\tilde{z}(\tau)e^{j\omega_c \tau} \right\} d\tau +$$

$$\int \tfrac{1}{2}\mathrm{Re}\left\{ 2\tilde{h}(t-\tau)e^{j\omega_c(t-\tau)}\tilde{z}(\tau)^* e^{-j\omega_c \tau} \right\} d\tau$$

$$\boxed{= \mathrm{Re}\left\{ \int \tilde{h}(t-\tau)\tilde{z}(\tau)d\tau \, e^{j\omega_c t} \right\}}$$

NOKIA

(cont'd)

and hence the complex envelope of the output w(t) is

$$\boxed{\tilde{w}(t) = \int \tilde{h}(t-\tau)\tilde{z}(\tau)d\tau} \qquad (22)$$

Note the use of number 2 in defining
the complex envelope of *filter impulse response*

[However, in simulations this is usually not critical;
it is only constant scaling for signals and noise]

Bandpass random signals and noise

Assume wide sense stationary (WSS) BP random signal

$$n(t) = \mathrm{Re}\left\{\tilde{n}(t)e^{j\omega_c t}\right\} = n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t) \tag{23}$$

or for simplicity

$$n(t) = x(t)\cos(\omega_c t) - y(t)\sin(\omega_c t) \tag{24}$$

Autocorrelation function

$$\begin{aligned} R_n(\tau) &= \overline{n(t)n(t+\tau)} \\ &= \tfrac{1}{2}\mathrm{Re}\left\{\overline{\tilde{n}(t)^*\tilde{n}(t+\tau)}e^{j\omega_c\tau}\right\} + \cancel{\tfrac{1}{2}\mathrm{Re}\left\{\overline{\tilde{n}(t)\tilde{n}(t+\tau)}e^{j\omega_c\tau + j2\omega_c t}\right\}} \end{aligned} \tag{25}$$

Due to WSS => second term must be ZERO

Bandpass random signals and noise (2)

Define

$$R_{\tilde{n}}(\tau) = \overline{\tilde{n}(t)^* \tilde{n}(t+\tau)}$$

$$= x(t)x(t+\tau) + y(t)y(t+\tau) + j[x(t)y(t+\tau) - y(t)x(t+\tau)$$ (26)

$$= R_x(\tau) + R_y(\tau) + j[R_{xy}(\tau) - R_{yx}(\tau)]$$

Then

$$\boxed{R_n(\tau) = \tfrac{1}{2}\mathrm{Re}\left\{R_{\tilde{n}}(\tau)e^{j\omega_c\tau}\right\}}$$ (27)

WSS condition gives

$$\overline{\tilde{n}(t)\tilde{n}(t+\tau)} = R_x(\tau) - R_y(\tau) + j[R_{xy}(\tau) + R_{yx}(\tau)] = 0$$ (28)

which gives general properties for the inphase and quadrature correlations

Bandpass random signals and noise (3)

$$\boxed{\begin{aligned} R_x(\tau) &= R_y(\tau) \\ R_{xy}(\tau) &= -R_{yx}(\tau) \end{aligned}} \tag{29}$$

and hence

$$\boxed{R_{\tilde{n}}(\tau) = 2R_x(\tau) + 2jR_{xy}(\tau)} \tag{30}$$

Note that the autocorrelation of the complex envelope may be complex (but the spectrum is real!)

Fourier transforming (27) we get

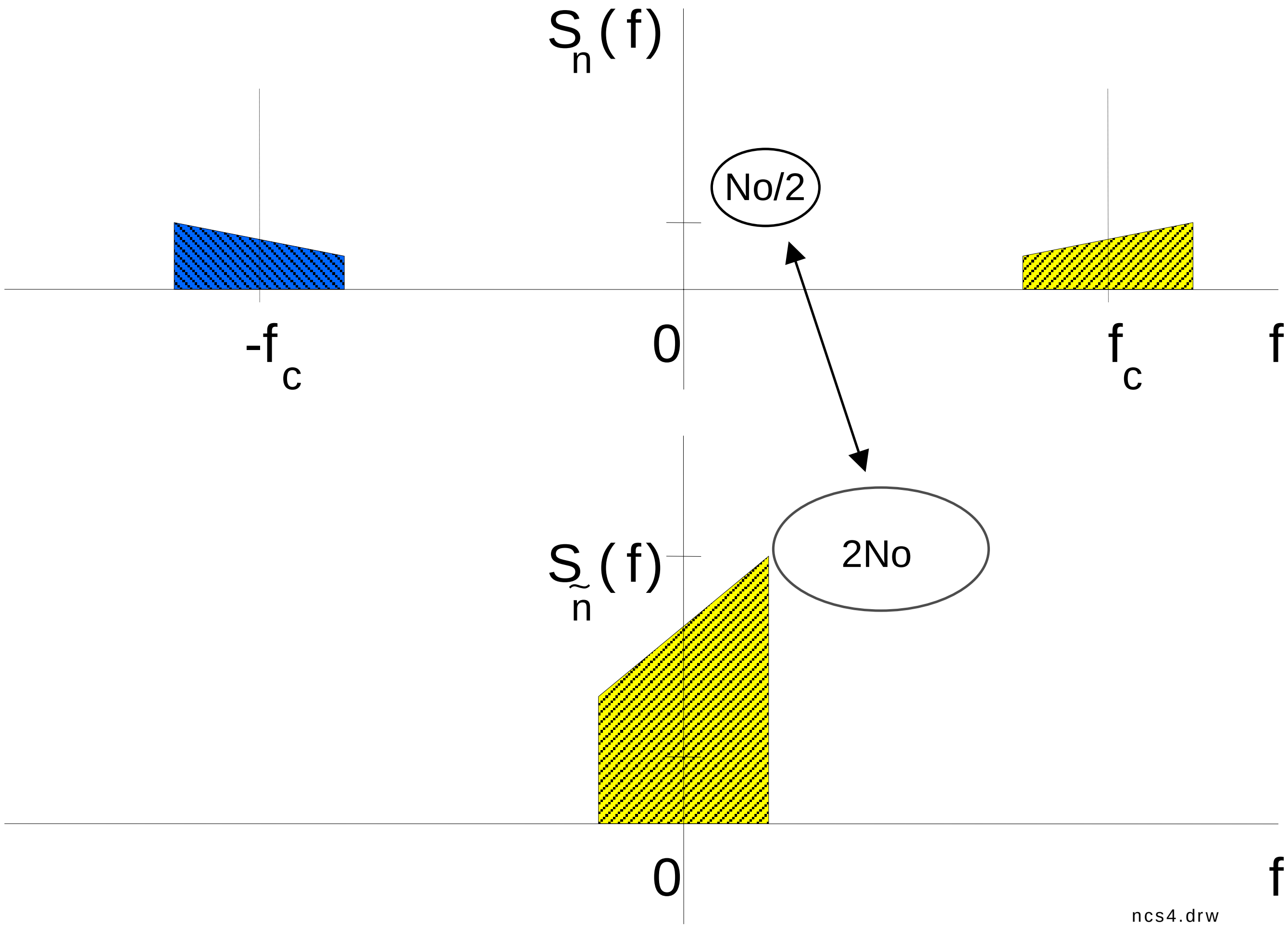
$$S_n(f) = \tfrac{1}{4}[S_{\tilde{n}}(f - f_c) + S_{\tilde{n}}(-f - f_c)] \tag{31}$$

As before we find low-pass spectrum

(U(f) is the "unilateral step function")

Bandpass random signals and noise (4)

$$S_{\tilde{n}}(f) = 4S_n(f + f_c)U(f + f_c) \qquad (32)$$



Spectra of quadrature components x(t) and y(t)

From (30) we get

$$R_x(\tau) = \tfrac{1}{4}[R_{\tilde{n}}(\tau) + R_{\tilde{n}}(\tau)^*]$$ (33)

$$R_{xy}(\tau) = \tfrac{1}{4j}[R_{\tilde{n}}(\tau) - R_{\tilde{n}}(\tau)^*]$$

and **F-transform** gives

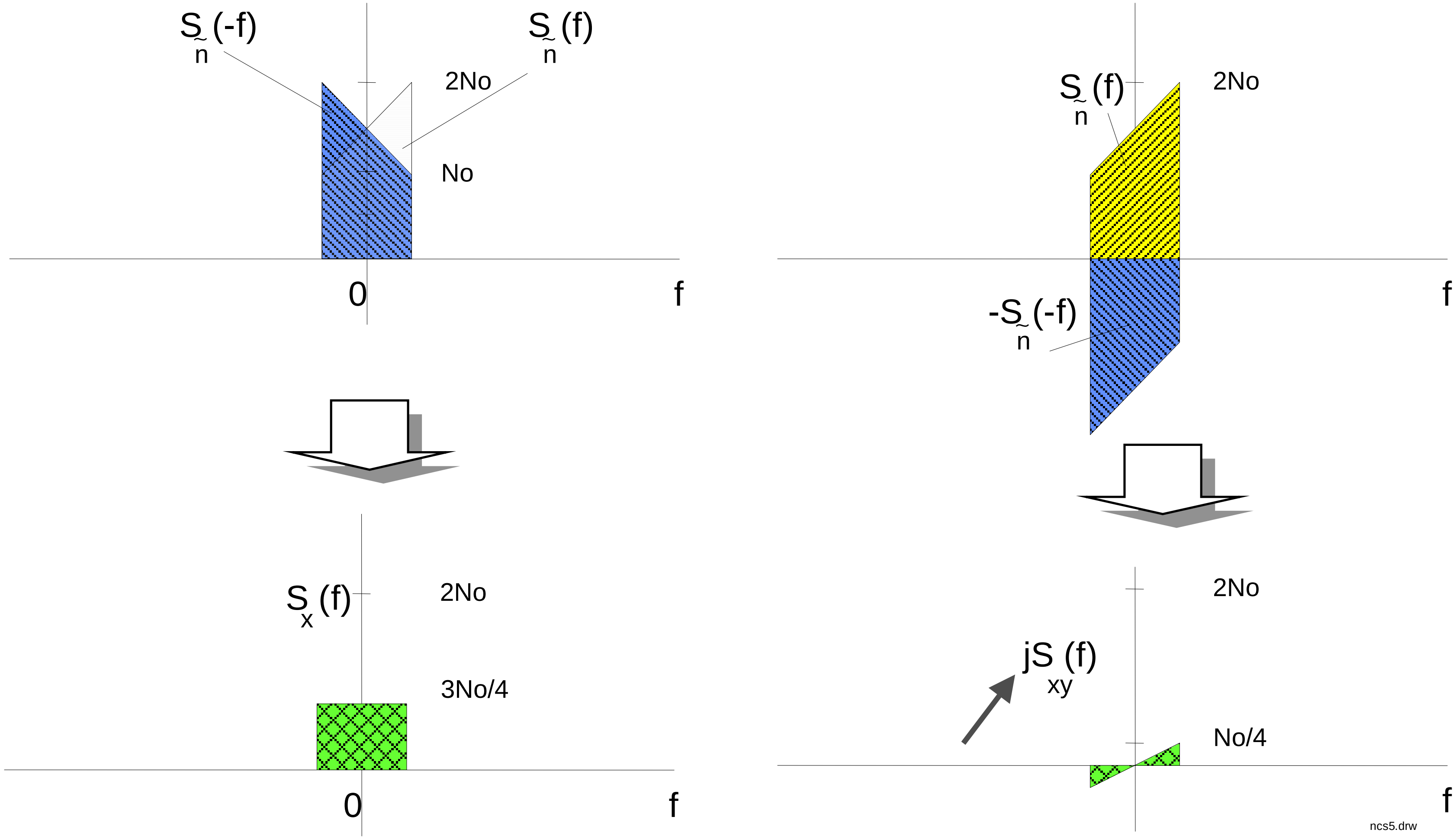
(34)

$$S_x(f) = \tfrac{1}{4}[S_{\tilde{n}}(f) + S_{\tilde{n}}(-f)]$$

$$S_{xy}(f) = \tfrac{1}{4j}[S_{\tilde{n}}(f) - S_{\tilde{n}}(-f)]$$

[see following picture]

Spectra and cross-spectra of low-pass components x and y



Conclusions about low-pass spectra

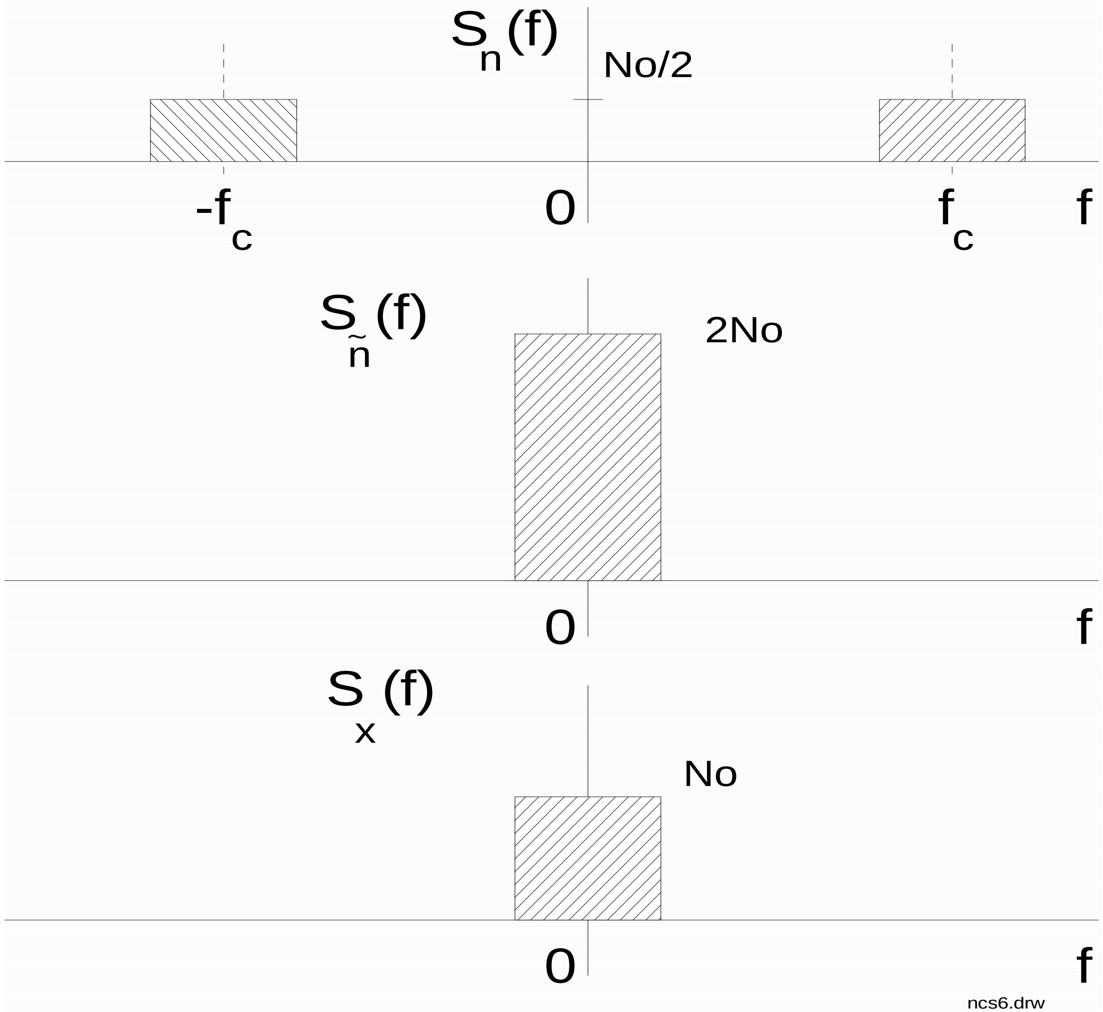
Note that if BP-spectrum is symmetric (ref f_c) then

$$
\begin{aligned}
&S_{xy}(f) \equiv 0 \\
&R_{xy}(\tau) \equiv 0 \\
&S_x(f) = \tfrac{1}{2} S_{\tilde{n}}(f) = 2S_n(f + f_c)U(f) \\
&R_x(\tau) = \tfrac{1}{2} R_{\tilde{n}}(\tau)
\end{aligned}
\qquad (35)
$$

So

- low-pass components are uncorrelated at any t
- low-pass spectrum is twice the (two-sided) BP spectrum
- Hey, keep eye on those damned "engineer's 2s"

Symmetric spectra



Power considerations

Power of the BP-process

$$P_n = R_n(0) = \sigma^2 \tag{36}$$

From (27) we have

$$R_{\tilde{n}}(0) = 2R_n(0) = 2\sigma^2 \tag{37}$$

From (29), (30) and (37) we conclude (real variables!)

$$\begin{aligned} &R_{xy}(0) = 0 \\ &R_x(0) = \tfrac{1}{2} R_{\tilde{n}}(0) = \sigma^2 = P_x = P_y \end{aligned} \tag{38}$$

Remarks

- Power of each component is the same as total BP power (general result)
- In general case inphase and quadrature components are uncorrelated at the SAME time instant (but not necessarily if $t \neq 0$)
- For Gaussian random process samples taken at the SAME instant are then independent
- Only for *symmetric* BP Gaussian process are also x(t) and y(t) independent Gaussian processes (at any time)

So, if you are dealing with **unsymmetric** cases
you might consider including correlation into your model

Complex white Gaussian process

Assume that

- BP spectrum is constant No/2 (two-sided), Gaussian
- ideal rectangular filtering, bandwidth W
- system filtering much narrower than W

then the complex noise envelope is characterized (may be approximated as)

$$
\begin{aligned}
S_{\tilde{n}}(f) &= 2N_o, \quad |f| \le W/2 \\
R_{\tilde{n}}(\tau) &= 2N_o\delta(\tau)
\end{aligned}
\tag{39}
$$

Complex white Gaussian process (2)

and the joint density function for x(t) and y(t) is

$$p_{xy}(x,y) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{x^2+y^2}{2\sigma^2}\right\} \qquad (40)$$

where σ^2 is the filtered noise power (e.g. N_oW). This can also be written in the form:

$$p_{\tilde{n}}(\tilde{n}) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{\left|\tilde{n}\right|^2}{2\sigma^2}\right\} \qquad (41)$$

EXTENSION RESULT:

General N-dimensional complex Gaussian

N-dimensional complex Gaussian density

Assume zero mean and stationarity

$$E[\tilde{\mathbf{x}}] = 0 \qquad (42)$$

$$E[\tilde{\mathbf{x}}\tilde{\mathbf{x}}^{T*}] \equiv 2\Lambda_{\tilde{\mathbf{x}}}$$

$$E[\tilde{\mathbf{x}}\tilde{\mathbf{x}}^{T}] = 0$$

where bold x means (column) vector, T stands for transpose and Λ for covariance matrix.

N-dimensional complex Gaussian density (2)

Probability density function for these joint complex Gaussian variables is

$$p_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}) = \frac{1}{(2\pi)^N \det\left|\Lambda_{\tilde{\mathbf{x}}}\right|} \exp\left\{-\tfrac{1}{2}\tilde{\mathbf{x}}^{T*}\Lambda_{\tilde{\mathbf{x}}}^{-1}\tilde{\mathbf{x}}\right\} \tag{43}$$

where

$$\tilde{\mathbf{x}}^T = \left[\tilde{x}_1, \tilde{x}_2, \ldots \tilde{x}_{N-1} \tilde{x}_N\right] \tag{44}$$

$$\Lambda_{\tilde{\mathbf{x}}} = \begin{bmatrix} \sigma_1^2 & \tfrac{1}{2}\overline{\tilde{x}_1 \tilde{x}_2^{\,*}} & \ldots & \\ \tfrac{1}{2}\overline{\tilde{x}_2 \tilde{x}_1^{\,*}} & \sigma_2^2 & & \\ \vdots & & & \\ \vdots & & & \sigma_N^2 \end{bmatrix}$$

Channel modeling

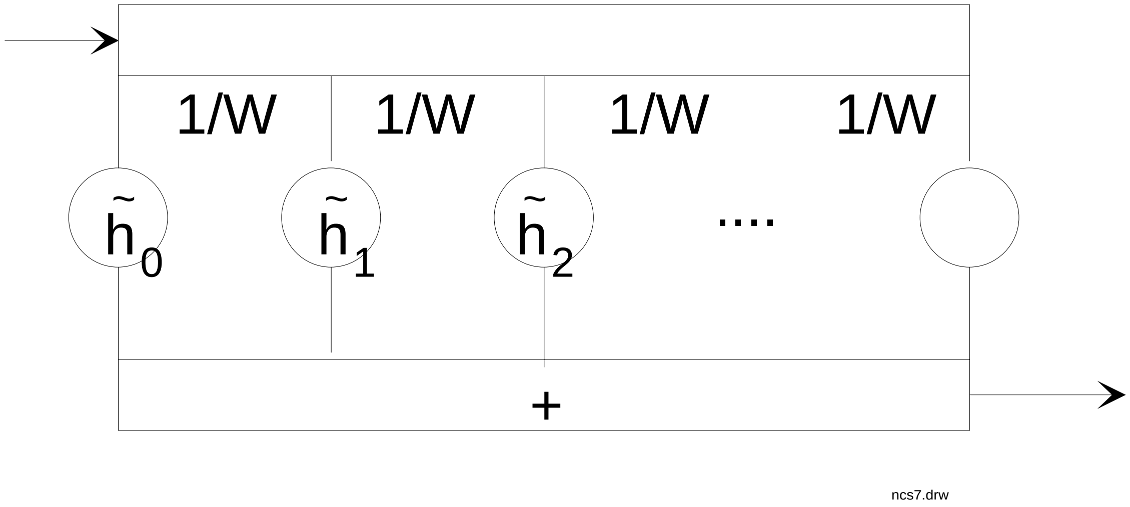
We use complex low-pass representation

- if the signal BP bandwidth is W only samples at rate W are needed
- system filtering may be represented using complex samples 1/W apart
- also channel may be represented with complex samples1/W apart

In practice somewhat denser sampling is preferable

Note also that time varying channel expands bandwidth (Doppler spread) and denser tapping is needed

Sampled channel



- Generally coefficients h_i are complex numbers (possibly time varying)
- Often channel is assumed "freezed" i.e. fixed for a number of transmitted symbols
- Often coefficients complex Gaussian

Examples

Fixed radio link channel

- normally only two taps needed
- freezing can be assumed
- main component may have nonzero mean i.e.

$$\begin{aligned}
\overline{\tilde{h}_0} &= \tilde{h}_{0m} = h_{0m} e^{j\varphi_o} \\
\overline{\tilde{h}_1} &= 0 \\
\tilde{h}_1 &= \left|\tilde{h}_1\right| e^{j\varphi_1}
\end{aligned} \qquad (45)$$

Note: magnitude of h_1 follows Rayleigh law in fading if it is zero mean complex Gaussian

Mobile radio channel

- normally a few taps needed (e.g. 5 or so)
- channel varies relatively fast compared to symbol lengths => Doppler spreads
- time variations are important => time varying coefficients often used in models
- in many cases all taps may be zero mean complex processes
- but also it may be possible that one or more taps have nonzero mean (specular reflections)

Modulated signals with complex envelopes

Many digital modulations can be represented in a general form with complex envelope $\tilde{z}(t)$

$$\tilde{z}(t) = \sum_n a_n u(t - nT) + j \sum_n b_n w(t - nT) \qquad (46)$$

where u(t) and w(t) are the basic pulse forms.

Example:

QPSK

a_n gets values $\pm 1 \pm j$ and b_n is zero

u(t) is, e.g., half Nyqvist pulse

Signal-to-noise ratio

Let us consider QPSK signal with u(t) rectangular pulse [0,T]

$$r(t) = v(t) + n(t) = \mathrm{Re}\left\{\sum_n a_n \sqrt{E} u(t - nT) e^{j\omega_c t}\right\} + \mathrm{Re}\left\{\tilde{n}(t) e^{j\omega_c t}\right\} \qquad (47)$$

Assume ideal T-integrator in the receiver

The output of the integrator at t=(k+1)T (compl.env)

$$\tilde{g}(kT) = \int_0^T \frac{1}{\sqrt{T}} a_k \sqrt{\tfrac{E}{T}} u(T - \tau) d\tau + \int_0^T \frac{1}{\sqrt{T}} \tilde{n}[(k+1)T - \tau] d\tau \qquad (48)$$

$$= a_k \sqrt{E} + \tilde{n}_1$$

Signal-to-noise ratio (2)

Signal power is now

$$\overline{\left|a_k\sqrt{E}\right|^2} = 2E \tag{49}$$

and noise power

$$\overline{\left|\tilde{n}_1\right|^2} = \tfrac{1}{T}\iint \overline{\tilde{n}[(k+1)T-\tau_1]\tilde{n}[(k+1)T-\tau_2]^*}\,d\tau_1 d\tau_2$$

$$= \tfrac{1}{T}\int_0^T 2N_o\,d\tau_1 = 2N_o \tag{50}$$

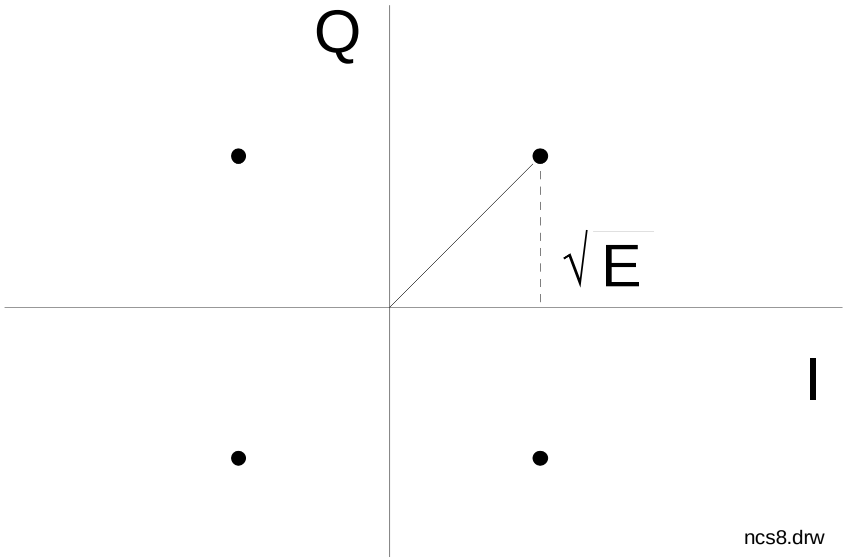
Note that variance of each noise component (I and Q) is now N_o.

Probability of error

Probality of symbol error (appr) is now

$$P_e = Q(\sqrt{\frac{E}{N_o}}) = Q(\sqrt{\frac{2E_b}{N_o}}) \tag{51}$$

where E_b is the bit energy and E is the real energy of one pulse.



Finally

- complex envelopes reduce simulation complexity
- complex envelopes make conceptually simple analysis of I-Q signals
- some care is needed when calculating signal-to-noise ratios (and may be signal-to-interference also) ["engineer’s 2’s"]
- WARNING: if system contains nonlinearities, one should watch what signal representations are valid (bandwidth expansion!)
- also fast time varying channels may require special treatment

References

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2. vanTrees H.,L.: Detection, Estimation and Modulation Theory, Part III. John Wiley & Sons, Inc., New York 1971, 626p.