Error probability of digital signaling

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Vector space representation of received signals

- In a flat fading channel with AWGN, the received complex envelope can be defined as:

\[
\tilde{r}(t) = g(t)\tilde{S}_i(t) + \tilde{n}(t)
\]

If \( f_m T << 1 \)

\[
\tilde{r}(t) = g\tilde{S}_i(t) + \tilde{n}(t)
\]

\( \tilde{S}_i(t) \) is M complex low-pass waveforms

\( \{ \tilde{S}_k(t) \} \) (k: from 0 to M-1)

\( g(t) \): a complex Gaussian random variable.

\( \tilde{n}(t) \): zero mean complex AWGN with a power spectral density (psd) of \( N_0 \) watts/Hz.

The vector can be expressed as:

\[
\tilde{r} = (\tilde{r}_0, \tilde{r}_1, ..., \tilde{r}_{N-1})
\]

\( \tilde{S}_i = (\tilde{s}_{i0}, \tilde{s}_{i1}, ..., \tilde{s}_{iN-1}) \)

\( \tilde{n} = (\tilde{n}_0, \tilde{n}_1, ..., \tilde{n}_{N-1}) \).
Probability of error

- the set M signal vectors \( \{ \tilde{S}_m \}_{m=0}^{M-1} \)
- The decision regions \( R_m = \{ \tilde{r} : \| \tilde{r} - g \tilde{s}_m \|^2 \leq \| \tilde{r} - g \tilde{s}_\hat{m} \|^2, \forall \hat{m} \neq m \} \)

\[
P(e | \tilde{s}_m) = 1 - \int_{R_m} p(\tilde{r} | g \tilde{s}_m) d\tilde{r}
\]

\[
P(e) = \frac{1}{M} \sum_{m=0}^{M-1} P(e | \tilde{S}_m)
\]
Upper bounds and lower bounds on error probability

- Upper bound can be obtained by computing the minimum squared Euclidean distance between any two-signal points \( \tilde{d}_{\min}^2 = \min_{n,m} \| \tilde{s}_n - \tilde{s}_m \|^2 \)
- the pair wise error probability
  \[ P(\tilde{S}_j, \tilde{S}_k) \leq Q\left( \sqrt{\frac{\alpha^2 \tilde{d}_{\min}^2}{4N_0}} \right) \]
  \[ P(e) \leq (M - 1)Q\left( \sqrt{\frac{\alpha^2 \tilde{d}_{\min}^2}{4N_0}} \right) \]

- lower bound on the error probability
  \[ P(e) \geq \frac{2}{M} Q\left( \sqrt{\frac{\alpha^2 \tilde{d}_{\min}^2}{4N_0}} \right) \]
Pairwise error probability

It can be defined for each pair of signal vectors in the signal constellation

the squared Euclidean distance \[ \tilde{d}_{jk}^2 = \| \tilde{S}_i - \tilde{S}_k \|^2 \]

So the pairwise error probability between the message vectors \( \tilde{S}_k \) and \( \tilde{S}_j \)

\[ P(\tilde{S}_j, \tilde{S}_k) = P(e | \tilde{S}_j) = P(e | \tilde{S}_k) = Q\left( \sqrt{\frac{\alpha^2 \tilde{d}_{jk}^2}{4N_0}} \right) \]
Bit and symbol error probabilities

- The symbol error probability is $P_M$ and the bit error probability is $P_b$

  \[
  \frac{P_M}{\log_2 M} \leq P_b \leq P_M
  \]

- **Gray codes**: symbol errors correspond to single bit errors where $k = \log_2 M$.

  \[
  P_b \approx \frac{P_M}{k}
  \]
Error probability of BPSK

- two signal waveforms are $s_1(t) = g(t)$ and $s_2(t) = -g(t)$.

$$s_1 = \sqrt{E_b}, s_2 = -\sqrt{E_b}$$

- When noise $n$ is present, the received signal from the demodulator is $r = s_1 + n$

$$p(r|s_1) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-\sqrt{E_b})^2}{N_0}}$$

$$p(r|s_2) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r+\sqrt{E_b})^2}{N_0}}$$

$$P(e|s_1) = \int_{-\infty}^{0} p(r|s_1)dr = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$P(e) = \frac{1}{2} P(e|s_1) + \frac{1}{2} P(e|s_2) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Fig. Conditional pdfs of two signals
Error probability of M-PSK

- The probability of M-ary symbol error, is just the probability that the received angle outsider the region $[-\frac{\pi}{M}, \frac{\pi}{M}]$

$$P_M(\gamma_s) = 1 - \int_{-\pi/M}^{\pi/M} p(\theta) d\theta$$
Error probability of M-PAM

- the Gray coded 8PAM system signal constellation

\[ P_i = 2Q\left(\frac{2\alpha^2 E_h}{N_0}\right) \quad P_o = Q\left(\frac{2\alpha^2 E_h}{N_0}\right) \]

\[ P_m = \frac{M-2}{M} P_i + \frac{2}{M} P_0 = 2(1 - \frac{1}{M}) Q\sqrt{\frac{2\alpha^2 E_h}{N_0}} \]
Error probability of M-QAM

- Rectangular QAM signal constellations “most frequently used in practice”
- Two PAM systems in quadrature
e.g. 16-QAM system can be treated as two independent Gray coded 4-PAM systems in quadrature, each operating with half the power of the 16-QAM system

The symbol error probability for each-PAM system is

$$P_{\sqrt{M}}(\gamma_s) = 2(1 - \frac{1}{\sqrt{M}})Q\left(\frac{\sqrt{6 \gamma_s}}{\sqrt{M} - 1} \frac{1}{2}\right)$$

the probability of error symbol reception in the M-QAM system is

$$P_M(\gamma_s) = 1 - (1 - P_{\sqrt{M}})^2$$
Error probability in a flat fading channel

(1/2)

■ When the channel experiences fading, the error probability must be averaged over the fading statistics.

■ The bit error probability of a fading channel is given by

\[ P_b = \int_0^\infty P_e(\gamma) p(\gamma) d\gamma \]

and the average symbol error probability is the symbol error probability over the

\[ P_M = \int_0^\infty P_M(\gamma) p(\gamma) d\gamma \]

■ If the channel is Rayleigh faded.

\[ p(\gamma) = \frac{1}{\gamma_0} \exp\left(-\frac{\gamma}{\gamma_0}\right) \]

\[ \gamma_0 = \frac{E_b}{N_0} \mathbb{E}[a^2] = \frac{2\sigma^2 E_b}{N_0}, \quad \gamma_0 \] is the mean value of the SNR.

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Error probability in a flat fading channel (2/2)

- Rayleigh fading converts an exponential dependency of the bit error probability on the bit energy-to-noise ratio into an inverse linear one.

Bit error probability for BPSK and QPSK on an AWGN channel and a Rayleigh fading channel with AWGN

Bit error probability for M-QAM on an AWGN channel and a Rayleigh fading channel with AWGN
Doppler spread impacts

- For OFDM, if the channel is time-varying, inter-channel interference (ICI) is introduced due to a loss of sub-channel orthogonality.
- At low $\tilde{\gamma}_b$, additive noise dominates the performance so that the extra noise due to ICI has little effect. However, ICI dominates the performance and causes an error floor at large $\tilde{\gamma}_b$.

Figure: BER for 16-QAM OFDM on a Rayleigh fading channel with various Doppler frequencies
Error probability over frequency selective fading channel

- The frequency selective channel will lead to ISI in the receiver, which will cause signal distortion. **The result is irreducible bit error probability floor, which means increasing the signal energy does not have any effect on the bit error probability.**

- This irreducible error floor can also due to the random FM caused by the time-varying Doppler spread, which in turn is due to the motion of the mobile.

- The computer simulations are the main tool used for analyzing frequency selective fading effects.
Reference

