Error probability of digital signaling

15, Jan. 2001

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Vector space representation of received signals

- In a flat fading channel with AWGN, the received complex envelope can be defined as:

$$\tilde{r}(t) = g(t)\tilde{s}_i(t) + \tilde{n}(t) \xrightarrow{\text{If } f_m T \ll 1} \tilde{r}(t) = g\tilde{s}_i(t) + \tilde{n}(t)$$

$\tilde{S}_i$(t) is M complex low-pass waveforms

{ $\tilde{S}_k$ (t)} (k: from 0 to M-1)

g(t): a complex Gaussian random variable.

$\tilde{n}(t)$: zero mean complex AWGN with a power spectral density (psd) of N_0 watts/Hz.

The vector can be expressed as:

$$\tilde{r} = g\tilde{s}_i + \tilde{n}$$

$$\tilde{r} = (\tilde{r}_0, \tilde{r}_1, \ldots, \tilde{r}_{N-1})$$

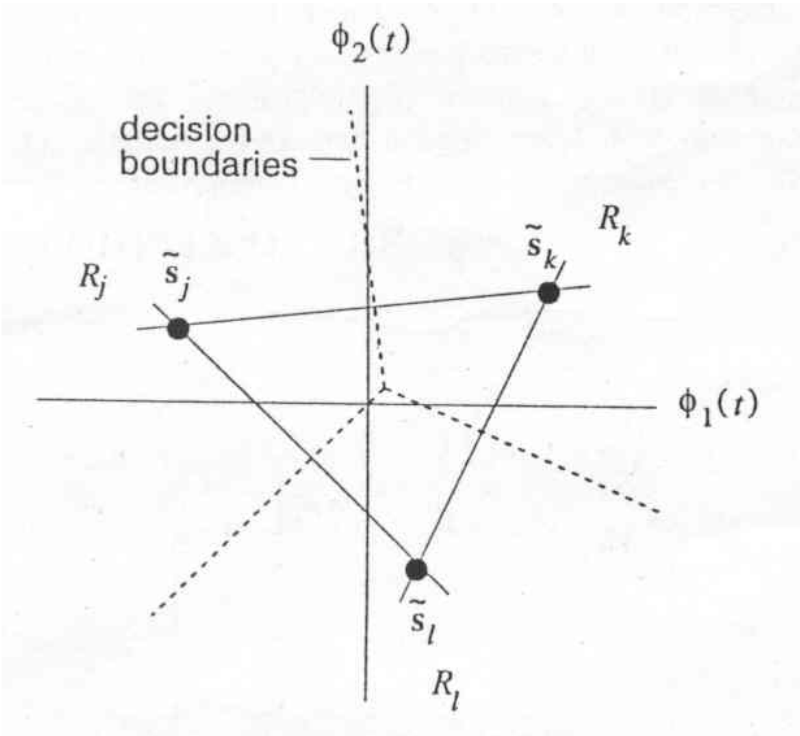
$$\tilde{s}_i = (\tilde{s}_{i0}, \tilde{s}_{i1}, \ldots, \tilde{s}_{iN-1})$$

$$\tilde{n} = (\tilde{n}_0, \tilde{n}_1, \ldots, \tilde{n}_{N-1}).$$

Probability of error

- the set M signal vectors $\left\{\tilde{S}_m\right\}_{m=0}^{M-1}$
- The decision regions $R_m = \left\{\tilde{r} : \left\|\tilde{r} - g\tilde{s}_m\right\|^2 \le \left\|\tilde{r} - g\tilde{s}_{\hat{m}}\right\|^2, \forall \hat{m} \ne m\right\}$
- $P(e|\tilde{s}_m) = 1 - \int_{R_m} p(\tilde{r}|g\tilde{s}_m)d\tilde{r}$

$$P(e) = \frac{1}{M}\sum_{m=0}^{M-1} P(e|\tilde{S}_m)$$



Upper bounds and lower bounds on error probability

- Upper bound can be obtained by computing the minimum squared Euclidean distance between any two-signal points $\tilde{d}^2_{\min} = \min_{n,m} \|\tilde{s}_n - \tilde{s}_m\|^2$

the pair wise error probability ,

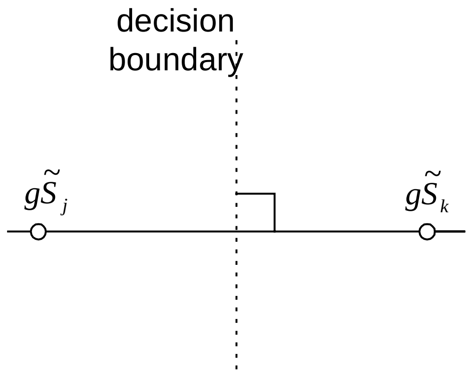
$$P(\tilde{S}_j, \tilde{S}_k) \le Q\left(\sqrt{\frac{\alpha^2 \tilde{d}^2_{\min}}{4N_0}}\right) \qquad P(e) \le (M-1)Q\left(\sqrt{\frac{\alpha^2 \tilde{d}^2_{\min}}{4N_0}}\right)$$

- lower bound on the error probability

$$P(e) \ge \frac{2}{M} Q\left(\sqrt{\frac{\alpha^2 \tilde{d}^2_{\min}}{4N_0}}\right)$$

Pairwise error probability

It can be defined for each pair of signal vectors in the signal constellation



the squared Euclidean distance $\tilde{d}_{jk}^{2} = \left\| \tilde{S}_i - \tilde{S}_k \right\|^2$

So the pairwise error probability between the message vectors

$\tilde{S}_k$ and $\tilde{S}_j$

$$P(\tilde{S}_j, \tilde{S}_k) = P(e|\tilde{S}_j) = P(e|\tilde{S}_k) = Q\left(\sqrt{\frac{\alpha^2 \tilde{d}_{jk}^{2}}{4N_0}}\right)$$

Bit and symbol error probabilities

- The symbol error probability is P_M and the bit error probability is P_b

$$\frac{P_M}{\log_2 M} \le P_b \le P_M$$

- **Gray codes** :symbol errors correspond to single bit errors where $k = \log_2 M$.

$$P_b \approx \frac{P_M}{k}$$

Error probability of BPSK

- two signal waveforms are $s_1(t)=g(t)$ and $s_2(t)= -g(t)$.

$$s_1 = \sqrt{E_b}, s_2 = -\sqrt{E_b}$$

- When noise n is present, The received signal from the demodulator is $r = s_1+n$

$$p(r|s_1) = \frac{1}{\sqrt{\pi N_0}} e^{-(r-\sqrt{E_b})^2/N_0} \_AND\_ p(r|s_2) = \frac{1}{\sqrt{\pi N_0}} e^{-(r+\sqrt{E_b})^2/N_0}$$

$$P(e|s_1) = \int_{-\infty}^{0} p(r|s_1)dr = Q(\sqrt{\frac{2E_b}{N_0}})$$

$$P(e) = \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) = Q(\sqrt{\frac{2E_b}{N_0}})$$

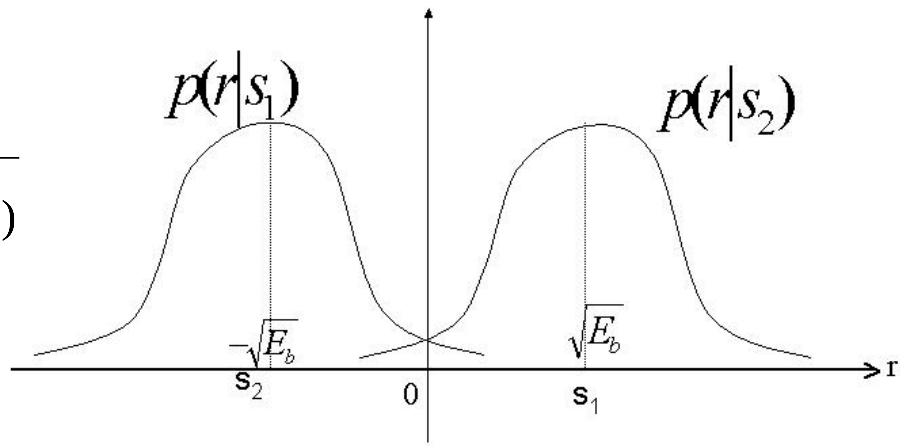
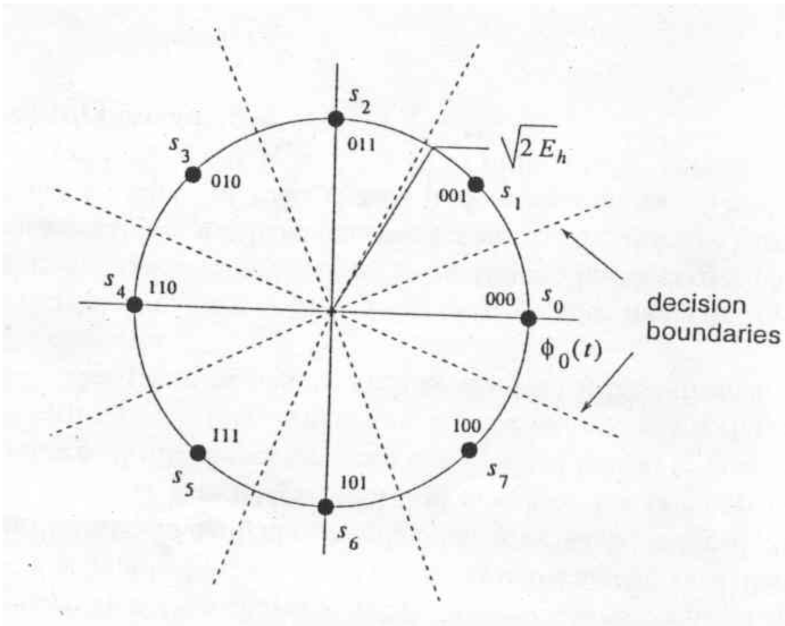


Fig. Conditional pdfs of two signals

Error probability of M-PSK

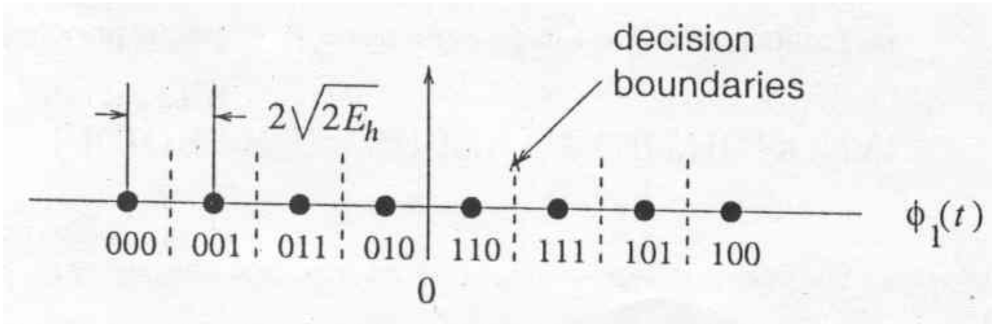
■ The probability of M-ary symbol error, is just the probability that the received angle outsider the region [- π /M, π /M]

$$P_M(\gamma_s) = 1 - \int_{-\pi/M}^{\pi/M} p(\theta) d\theta$$



Error probability of M-PAM

- the Gray coded 8PAM system signal constellation



$$P_i = 2Q\left(\frac{2\alpha^2 E_h}{N_0}\right) \qquad P_o = Q\left(\frac{2\alpha^2 E_h}{N_0}\right)$$

$$P_m = \frac{M-2}{M} P_i + \frac{2}{M} P_0 = 2(1-\frac{1}{M})Q\left(\sqrt{\frac{2\alpha^2 E_h}{N_0}}\right)$$

Error probability of M-QAM

- Rectangular QAM signal constellations "most frequently used in practice"
- two -PAM systems in quadrature

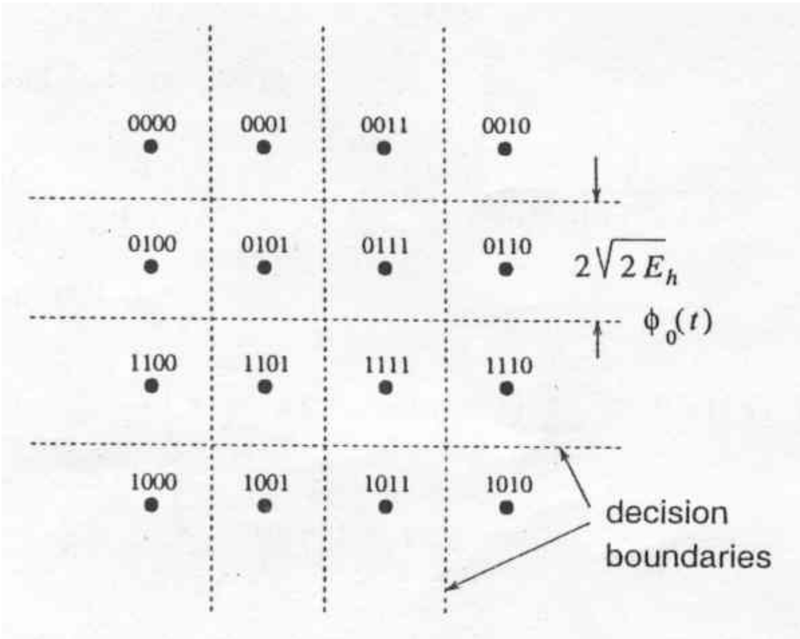
e.g.16-QAM system can be treated as two independent Gray coded 4-PAM systems in quadrature, each operating with half the power of the 16-QAM system

The symbol error probability for each-PAM system is

$$P_{\sqrt{M}}(\gamma_s) = 2(1 - \frac{1}{\sqrt{M}})Q\left(\sqrt{\frac{6}{M-1}\frac{\gamma_s}{2}}\right)$$

the probability of error symbol reception in the M-QAM system is

$$P_M(\gamma_s) = 1 - (1 - P_{\sqrt{M}})^2$$



Error probability in a flat fading channel (1/2)

- When the channel experiences fading, the error probability must be averaged over the fading statistics.

- The bit error probability of a fading channel is given by

$$P_b = \int_0^\infty P_e(\gamma) p(\gamma) d\gamma$$

and the average symbol error probability is the symbol error probability over the $p(\gamma)$

$$P_M = \int_0^\infty P_M(\gamma) p(\gamma) d\gamma$$

- If the channel is Rayleigh faded.

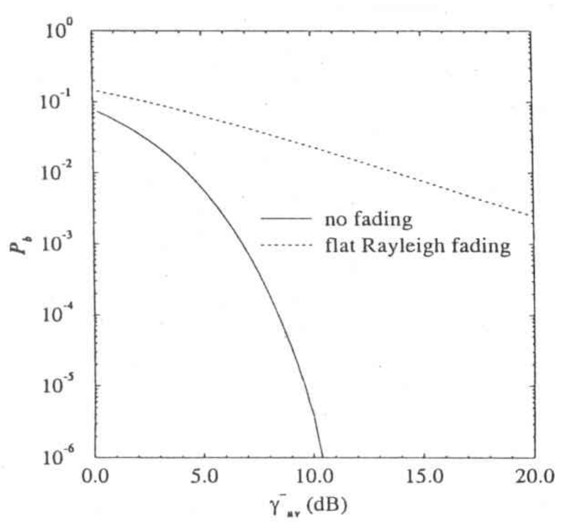
$$p(\gamma) = \frac{1}{\gamma_0} \exp(-\frac{\gamma}{\gamma_0})$$

$$\gamma_0 = \frac{E_b}{N_0} E\left[a^2\right] = \frac{2\sigma^2 E_b}{N_0},$$

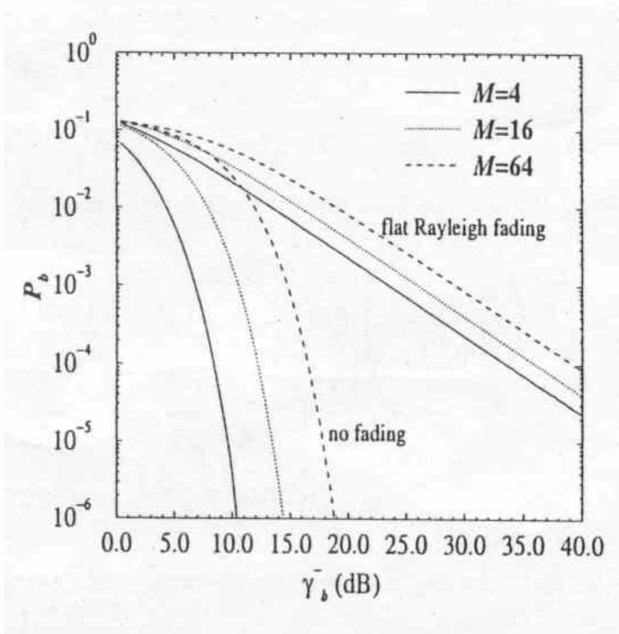
is the mean value of the SNR.

Error probability in a flat fading channel
(2/2)

- **Rayleigh fading converts an exponential dependency of the bit error probability on the bit energy-to-noise ratio into an inverse linear one**.



Bit error probability for BPSK and QPSK on an AWGN channel and a Rayleigh fading channel with AWGN



Bit error probability for M-QAM on an AWGN channel and a Rayleigh fading channel with AWGN

Doppler spread impacts

- For OFDM, if the channel is time-varying, inter-channel interference (ICI) is introduced due to a loss of sub-channel orthogonality
- At low $\tilde{\gamma}_b$, additive noise dominates the performance so that the extra noise due to ICI has little effect. However, ICI dominates the performance and causes an **error floor** at large $\tilde{\gamma}_b$.

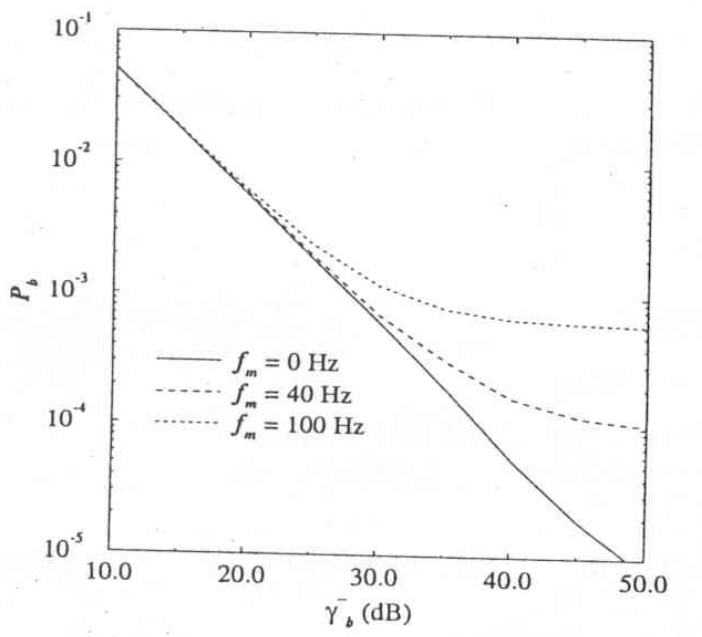


Figure: BER for 16-QAM OFDM on a Rayleigh fading channel with various Doppler frequencies

Error probability over frequency selective fading channel

- The frequency selective channel will lead to ISI in the receiver, which will cause signal distortion. **The result is irreducible bit error probability floor, which means increasing the signal energy does not have any effect on the bit error probability.**

- This irreducible error floor can also due to the random FM caused by the time-varying Doppler spread, which in turn is due to the motion of the mobile.

- The computer simulations are the main tool used for analyzing frequency selective fading effects

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