

## S-72.340 INFORMATION THEORY

### Tutorial 1, September 22, 2004, Problems

1. The sample space  $S$  of an random experiment is given by

$$S = \{a, b, c, d\}$$

with probabilities  $P(a) = 0.2$ ,  $P(b) = 0.3$ ,  $P(c) = 0.4$ ,  $P(d) = 0.1$ .

Let  $A$  denote event  $\{a, b\}$ ,  $B$  the event  $\{b, c, d\}$ . Determine the following probabilities  $P(A)$ ,  $P(B)$ ,  $P(A^c)$ ,  $P(A \cup B)$  and  $P(A \cap B)$ .

2. Two events  $A$  and  $B$  are said to be independent, if and only if  $P(A \cap B) = P(A)P(B)$ . The conditional probability of an event  $A$  given event  $B$ , denoted by  $P(A|B)$ , is defined as  $P(A|B) = P(A \cap B)/P(B)$ .

Show that

- a) If events  $A$  and  $B$  are independent, then events  $A$  and  $B^c$  and events  $A^c$  and  $B^c$  are also independent.
- b) If events  $A$  and  $B$  are independent, then  $P(A|B) = P(A)$
- c)  $P(A|B) = P(B|A)P(A)/P(B)$ . (Bayes Rule)
- e)  $P(A \cap B|C) = P(A|B \cap C)P(B|C)$
3. Urn  $A$  contains one red ball and one white ball. Urn  $B$  contains one red ball and  $n$  white balls. Urn is selected at random and one ball is drawn.
- a) What is probability that ball is red?
- b) Suppose drawn ball was red. What is probability that selected urn was urn  $A$ ?
4. (Cover 2.1) A fair coin is flipped until the first head occurs. Let  $X$  denote the number of flips required.

(a) Find the entropy  $H(X)$  in bits. The following expressions (Geometric series) may be useful:

$$\sum_{n=1}^{\infty} r^n = r/(1-r),$$

$$\sum_{n=1}^{\infty} nr^n = r/(1-r)^2.$$

(b) A random variable  $X$  is drawn according to this distribution. Find an "efficient" sequence of yes-no questions of the form, "Is  $X$  contained in the set  $S$ ?" Compare  $H(X)$  to the expected number of questions required to determine  $X$ .

5. Distributions of random variables  $X$  and  $Y$  are  $P(X = 0) = P(X = 1) = 1/2$ ,  $P(Y = 0) = 1/3$  and  $P(Y = 1) = 2/3$ .  $X$  and  $Y$  are independent (i.e  $P(X = i, Y = j) = P(X = i)P(Y = j)$ ,  $i, j \in \{0, 1\}$ ). Random variable  $Z$  is defined as  $Z = (X + Y) \bmod 2$ .

a) Find  $H(X)$ ,  $H(Y)$  and  $H(Z)$ .

b) Find  $H(X, Y)$ ,  $H(X, Z)$  and  $H(Y, Z)$ .