S-72.340 INFORMATION THEORY

Exercise 11, November 29, 2000 **Solutions**

1. The input X has the pdf

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-x^2/2\sigma_X^2} .$$

The random variables X and G are independent, and we get:

$$\begin{aligned}
H(X|G) &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,n)\log p(x|n)\,dx\,dn \\
&= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x)p(n)\log p(x)\,dx\,dn \\
&= -\int_{-\infty}^{\infty} p(n)\,dn \cdot \int_{-\infty}^{\infty} p(x)\log p(x)\,dx \\
&= -\int_{-\infty}^{\infty} p(x)\log p(x)\,dx = H(X) \\
&= -\frac{1}{\sqrt{2\pi}\sigma_X}\int_{-\infty}^{\infty} e^{-x^2/2\sigma_X^2}\left(\frac{-x^2}{2\sigma_X^2}\log e - \log\sqrt{2\pi\sigma_X^2}\right)dx \\
&= \frac{\log e}{2\sigma_X^2}\cdot\frac{1}{\sqrt{2\pi}\sigma_X}\int_{-\infty}^{\infty} x^2 e^{-x^2/2\sigma_X^2}\,dx + \frac{1}{2}\log(2\pi\sigma_X^2)\cdot\frac{1}{\sqrt{2\pi}\sigma_X}\int_{-\infty}^{\infty} e^{-x^2/2\sigma_X^2}\,dx \\
&= \frac{\log e}{2\sigma_X^2}\cdot\sigma_X^2 + \frac{1}{2}\log(2\pi\sigma_X^2) = \frac{1}{2}\log(2\pi e\,\sigma_X^2)\,.
\end{aligned}$$

2. We use the relation
$$I(X;Y) = H(Y) - H(Y|X)\,.$$
Since X and G are zero-mean gaussian random variables, also Y is gaussian with zero mean, and
$$H(Y) = \frac{1}{2}\log\Big(2\pi e(\sigma_X^2 + \sigma_n^2)\Big)\,.$$
Since $Y = X + G$, the conditional pdf $p(y|x)$ is given by
$$p(y|x) = p_G(y-x) = \frac{1}{\sqrt{2\pi}\sigma_n} e^{-(y-x)^2/2\sigma_n^2}\,.$$
The conditional entropy $H(Y|X)$:
$$\begin{aligned}
H(Y|X) &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\log p(y|x)\,dx\,dy \\
&= -\int_{-\infty}^{\infty} p(x)\log e \int_{-\infty}^{\infty} p(y|x)\ln\left(\frac{1}{\sqrt{2\pi}\sigma_n} e^{-(y-x)^2/2\sigma_n^2}\right) dy\,dx \\
&= \int_{-\infty}^{\infty} p(x)\log e\left[\int_{-\infty}^{\infty} p_G(y-x)\left(\ln(\sqrt{2\pi}\sigma_n) + \frac{(y-x)^2}{2\sigma_n^2}\right) dy\right] dx \\
&= \int_{-\infty}^{\infty} p(x)\log e\left[\ln(\sqrt{2\pi}\sigma_n) + \frac{1}{2\sigma_n^2}\sigma_n^2\right] dx \\
&= \left[\log(\sqrt{2\pi}\sigma_n) + \frac{1}{2}\log e\right]\int_{-\infty}^{\infty} p(x)\,dx = \frac{1}{2}\log(2\pi e\,\sigma_n^2) = H(G)\,.
\end{aligned}$$
So, the average mutual information becomes:
$$\begin{aligned}
I(X;Y) &= H(Y) - H(Y|X) \\
&= \frac{1}{2}\log\Big(2\pi e(\sigma_X^2 + \sigma_n^2)\Big) - \frac{1}{2}\log(2\pi e\,\sigma_n^2) = \frac{1}{2}\log\left(1 + \frac{\sigma_X^2}{\sigma_n^2}\right).
\end{aligned}$$

3. The total channel is clearly a BSC, because all the component channels are BSC's, and thus it makes no difference to the probability of error if the input symbol is 0 or 1. An error occurs if an odd number of channels produces an error. The probability of this is

$$\epsilon_n = P_{\text{odd}} = \sum_{k\,\text{odd}} \binom{n}{k} \epsilon^k (1-\epsilon)^{n-k} \,.$$

The probability that an even number of channels produces an error is naturally given by

$$P_{\text{even}} = \sum_{k\,\text{even}} \binom{n}{k} \epsilon^k (1-\epsilon)^{n-k} \,,$$

and the following equation holds:

$$P_{\text{odd}} + P_{\text{even}} = 1 \,. \tag{1}$$

We obtain another relation between P_{odd} and P_{even} by considering their difference:

$$\begin{aligned}
P_{\text{even}} - P_{\text{odd}} &= \sum_{k\,\text{even}} \binom{n}{k} \epsilon^k (1-\epsilon)^{n-k} - \sum_{k\,\text{odd}} \binom{n}{k} \epsilon^k (1-\epsilon)^{n-k} \\
&= \sum_{k\,\text{even}} \binom{n}{k} (-\epsilon)^k (1-\epsilon)^{n-k} + \sum_{k\,\text{odd}} \binom{n}{k} (-\epsilon)^k (1-\epsilon)^{n-k} \qquad (2) \\
&= (1-\epsilon-\epsilon)^n = (1-2\epsilon)^n \,.
\end{aligned}$$

Solving from (1) and (2), we get

$$\epsilon_n = P_{\text{odd}} = \frac{1}{2}(1-(1-2\epsilon)^n) \,.$$

Since the total channel is a BSC, its capacity is given by

$$C(n) = 1 - H(\epsilon_n) \,,$$

where H denotes the binary entropy function. If $n \to \infty$, $\epsilon_n \to 1/2$, and

$$C = \lim_{n\to\infty} C(n) = 1 - H(\frac{1}{2}) = 0 \,.$$

4. If there is a 1 at the input, there will be a 1 at the output, too. If there is a 0 at the input, the probability that there is a 1 at the output is given by

$$\begin{aligned}P(Y=1|X=0) &= \epsilon + (1-\epsilon)\epsilon + \cdots + (1-\epsilon)^{n-1}\epsilon \\ &= \epsilon \cdot \sum_{i=0}^{n-1}(1-\epsilon)^i = \epsilon \cdot \frac{1-(1-\epsilon)^n}{1-(1-\epsilon)} = 1-(1-\epsilon)^n\,.\end{aligned}$$

So, there was an error in the problem setting. This was because this problem was taken straight (and blindly) from a book (Proakis) where Z channel, and especially the parameter ϵ, has been defined differently. Anyway, the resulting channel is equivalent to a Z channel with

$$\epsilon_1 = 1-(1-\epsilon)^n\,.$$

If $n \to \infty$, then clearly $\epsilon_1 \to 1$, and thus the output of the limiting channel consists only of 1's.

5. At first we must calculate the state probabilities $P(x_1)$ and $P(x_2)$. The standard way to do this is to solve them as a row vector P from the equation $P = P\mathbf{\Pi}$, where $\mathbf{\Pi}$ is the transition probability matrix of the Markov chain. In addition, the normalizing condition $P(x_1)+P(x_2)=1$ is needed. In this case,

$$\mathbf{\Pi} = \begin{bmatrix}0.8 & 0.2\\ 0.3 & 0.7\end{bmatrix}$$

and we get $P(x_1)=0.6$ and $P(x_2)=0.4$. The source entropy is

$$\begin{aligned}H(X) &= P(x_1)[-P(x_1|x_1)\log P(x_1|x_1) - P(x_2|x_1)\log P(x_2|x_1)] \\ &\quad + P(x_2)[-P(x_1|x_2)\log P(x_1|x_2) - P(x_2|x_2)\log P(x_2|x_2)] \\ &= 0.6[-0.8\log 0.8 - 0.2\log 0.2] + 0.4[-0.3\log 0.3 - 0.7\log 0.7] \\ &\approx 0.7857\text{bits/letter.}\end{aligned}$$

If a binary DMS has the same output letter probabilities, its entropy is

$$H_{\mathrm{DMS}} = -0.6\log 0.6 - 0.4\log 0.4 \approx 0.971 \text{ bits/letter.}$$

The memory inherent in the Markov source reduces the information content of each output letter.