S-72.340 INFORMATION THEORY

Exercise 1, September 20, 2000 **Solutions**

1. $E_0 = E_1 \cup E_2 \cup E_3$

   a) $P(E_0) = P(E_1) + P(E_2) + P(E_3) = 0.75$.

   b) By De Morgan's law, $E_0^c = E_1^c \cap E_2^c \cap E_3^c$. If E_1, E_2 and E_3 are independent, so are their complements, and we get:

$$\begin{aligned} P(E_0) &= 1 - P(E_0^c) \\ &= 1 - P(E_1^c) \cdot P(E_2^c) \cdot P(E_3^c) \\ &= 1 - 0.75^3 \approx 0.578. \end{aligned}$$

   c) $P(E_0) = 0.25$.

2. a) If we define A to be the event that at least one double 6 occurs, then A^c is the event that none of the 10 throws yields a double 6. The probability that a double 6 occurs at a single throw is $(1/6)^2 = 1/36$.

$$P(A) = 1 - P(A^c) = 1 - (\frac{35}{36})^{10} \approx 0.246.$$

   b) $A =$the event that the first ball is white, $B =$the event that the second ball is white.

$$P(A \cap B) = P(A) \cdot P(B|A) = \frac{8}{12} \cdot \frac{7}{11} = \frac{14}{33} \approx 0.424.$$

3. By the definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} > \frac{P(A \cap B)}{P(A)} = P(B|A)\,.$$

4. $A =$the event that a patient is cured, $B =$ the event that a patient received a drug treatment. By the Bayes' rule,

$$\begin{aligned} P(B|A) &= \frac{P(A \cap B)}{P(A)} = \frac{P(A|B) \cdot P(B)}{P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c)} \\ &= \frac{0.75 \cdot 0.1}{0.75 \cdot 0.1 + 0.5 \cdot 0.9} \approx 0.143. \end{aligned}$$

5. The integral of $f_X(x)$ over the interval $(0,2)$ must be equal to 1:

$$\int_0^2 f_X(x)\,dx = k \cdot \int_0^2 (2x - x^2)\,dx = \cdots = \frac{4k}{3} = 1 \Rightarrow k = \frac{3}{4}.$$

The cdf $F_X(x)$ is defined as the probability that $X \leq x$. Clearly, $F_X(x) = 0$ if $x \leq 0$, and $F_X(x) = 1$ if $x \geq 2$. If $x \in (0,2)$,

$$F_X(x) = \frac{3}{4} \cdot \int_0^x (2u - u^2)\,du = \cdots = \frac{3}{4}x^2 - \frac{1}{4}x^3.$$

$$P(X > 1) = 1 - F_X(1) = 1 - \frac{1}{2} = \frac{1}{2}.$$

6. The Poisson distribution with parameter λ is defined as

$$p_X(k) = \frac{\lambda^k}{k!}e^{-\lambda}, \quad k = 0, 1, 2, \ldots$$

$$\begin{aligned} p_X(0) &= e^{-\lambda} \Rightarrow \lambda = -\ln p_X(0) \\ &= -\ln 0.0498 \approx 3 \end{aligned}$$

It is well known that the mean of the Poisson distribution (and also the variance) is equal to the parameter λ. So, $E(X) \approx 3$.

$$\begin{aligned} P(X \geq 3) = 1 - P(X < 3) &= 1 - \sum_{k=0}^{2} p_X(k) \\ &= 1 - \sum_{k=0}^{2} \frac{3^k}{k!}e^{-3} \approx 0.577. \end{aligned}$$