S-72.340 INFORMATION THEORY

Exercise 2, September 27, 2000

1. The Chebyshev inequality can be written in many different forms. One of these is the following:

   Let X be a random variable with mean μ and a finite variance σ^2. Then

   $$P(\,|\,X-\mu\,| \geq k\sigma\,) \leq \frac{1}{k^2}\,, \quad \text{where } k > 0.$$

   a) Show that

   $$P(\,\mu - \epsilon < X < \mu + \epsilon\,) \geq 1 - \frac{\sigma^2}{\epsilon^2}\,.$$

   b) Suppose that A is an event, and that the probability of A in a certain experiment is p. The experiment is repeated n times so that the individual experiments are independent. Let k denote the number of times that A occurs in n experiments. Show that, for any $\epsilon > 0$,

   $$\lim_{n\to\infty} P(\,|\,\frac{k}{n} - p\,| \leq \epsilon\,) = 1.$$

   This is the (weak) law of large numbers for repeated Bernoulli trials.

2. Let X be a geometrically distributed random variable; that is,

   $$p_X(k) = p(1-p)^{k-1}\,,\ k = 1, 2, 3, \ldots$$

   Find the entropy of X.

3. Determine the entropy $H(X)$ of the uniformly distributed random variable X with pdf

   $$f_X(x) = \begin{cases} a^{-1} & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

   for the following three cases: $a = 1$, $a = 4$ and $a = 1/4$.

4. The moment generating function of a random variable X is defined by

$$M_X(t) = \begin{cases} \sum_i e^{tx_i}\, p_X(x_i) & \text{(discrete case)} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx & \text{(continuous case)} \end{cases}$$

where t is a real variable. Find $M_X(t)$ if

$$f_X(x) = \frac{1}{\sqrt{32\pi}}\, e^{-(x+7)^2/32} \qquad -\infty < x < \infty\,.$$

5. Let Y be a uniform random variable over $(0,1)$. Let $F(x)$ be a function which has the properties of the cdf of a continuous random variable. Let $X = F^{-1}(Y)$. Show that the cdf of X is $F(x)$. Based on this result, how can you transform random numbers that are uniformly distributed over $(0,1)$ into random numbers from the Cauchy distribution with parameter 1? The pdf of this distribution is

$$f_X(x) = \frac{1}{\pi\,(\,1+x^2\,)} \qquad -\infty < x < \infty\,.$$

Try to solve as many problems as you can. Return your solutions by the time that the exercise begins. You can bring them to the exercise or put them into the 'return box' below the information board of our laboratory on the 2nd floor.