S-72.340 INFORMATION THEORY

Exercise 2, September 27, 2000 **Solutions**

1. a) Set $k\sigma = \epsilon$. Then $k = \epsilon/\sigma$, and by the Chebyshev inequality,

$$\begin{aligned} &P(\,|\,X-\mu\,| \geq \epsilon\,) \leq \frac{\sigma^2}{\epsilon^2} \\ \Leftrightarrow\, &1 - P(\,|\,X-\mu\,| < \epsilon\,) \leq \frac{\sigma^2}{\epsilon^2} \\ \Leftrightarrow\, &P(\,\mu-\epsilon < X < \mu+\epsilon\,) \geq 1 - \frac{\sigma^2}{\epsilon^2}\,. \end{aligned}$$

b) Define random variables $X_i\,, i = 1, \ldots, n$:

$$X_i = \begin{cases} 1 & \text{if } A \text{ occurs at the } i\text{th trial} \\ 0 & \text{otherwise} \end{cases}$$

For all i, $E(\,X_i\,) = p$ and $\sigma^2_{X_i} = pq$, where $q = 1 - p$. (Check this!) The random variables X_i are independent and identically distributed (i.i.d.). The sample mean $\bar{X_n}$ is given by

$$\bar{X_n} = \frac{X_1 + \cdots + X_n}{n} = \frac{k}{n}\,.$$

From the basic calculation rules for expectation and variance, it is easily obtained that

$$E(\,\bar{X_n}\,) = p \quad \text{and} \quad \sigma^2_{\bar{X_n}} = \frac{pq}{n}\,.$$

Further, $pq = p(1-p) \leq 1/4$. (Verify this!) So, by applying the result from 1a), we get

$$\begin{aligned} &P(\,|\,\frac{k}{n} - p\,| < \epsilon\,) = P(\,|\,\bar{X_n} - p\,| < \epsilon\,) \\ &\geq 1 - \frac{pq}{n\epsilon^2} \geq 1 - \frac{1}{4n\epsilon^2} \xrightarrow[n\to\infty]{} 1. \end{aligned}$$

Because a lower bound of the probability approaches the value 1 as $n \to \infty$, the limit of the probability must be equal to 1.

2. By the definition of entropy,

$$
\begin{aligned}
H(X) &= -\sum_{k=1}^{\infty} p_X(k) \cdot \log[\, p_X(k)\,] \\
&= -\sum_{k=1}^{\infty} p(1-p)^{k-1} \cdot \log[\, p(1-p)^{k-1}\,] \\
&= -p\log p \cdot \sum_{k=1}^{\infty}(1-p)^{k-1} - p\log(1-p) \cdot \sum_{k=1}^{\infty}(k-1)\cdot(1-p)^{k-1} \\
&= -p\log p \cdot S_1 - p\log(1-p) \cdot S_2\,. \\
S_1 &= \sum_{k=1}^{\infty}(1-p)^{k-1} = \frac{1}{p}\,. \\
S_2 &= \sum_{k=1}^{\infty}(k-1)\cdot(1-p)^{k-1} \\
&= \sum_{k=2}^{\infty}(k-1)\cdot q^{k-1} = q \cdot \sum_{k=2}^{\infty}(k-1)\cdot q^{k-2} \\
&= q \cdot \frac{d}{dq}\left\{\sum_{k=2}^{\infty} q^{k-1}\right\} = q \cdot \frac{d}{dq}\left\{\frac{q}{1-q}\right\} \\
&= \cdots = \frac{q}{(1-q)^2} = \frac{1-p}{p^2}\,. \\
\Rightarrow H(X) &= -\log p - \frac{1-p}{p}\log(1-p)\,.
\end{aligned}
$$

3. The (differential) entropy of X is given by

$$
\begin{aligned}
H(X) &= \int_{-\infty}^{\infty} f_X(x) \cdot \log[1/f_X(x)]\, dx \\
&= \int_{0}^{a} a^{-1} \cdot \log a \, dx = \log a\,.
\end{aligned}
$$

So, we get

$$
H(X) = \begin{cases} 0 & a = 1 \\ \log 4 & a = 4 \\ -\log 4 & a = 1/4 \end{cases}
$$

Note that the 'entropy' of a continuous random variable can be negative.

4. By the definition of the moment generating function of a continuous random variable,

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx$$
$$= \frac{1}{\sqrt{32\pi}} \int_{-\infty}^{\infty} e^{tx-(x+7)^2/32}\, dx\,.$$

The next thing to do is to complete the square in the exponent:

$$tx - \frac{(x+7)^2}{32} = \frac{-x^2 + (32t-14)x - 49}{32}$$
$$= \frac{-x^2 + (32t-14)x - (16t-7)^2}{32} + \frac{(16t-7)^2 - 49}{32}$$
$$= -\frac{[x-(16t-7)]^2}{32} + 8t^2 - 7t\,.$$

So, the moment generating function becomes:

$$M_X(t) = e^{8t^2-7t} \cdot \frac{1}{\sqrt{32\pi}} \int_{-\infty}^{\infty} e^{-[x-(16t-7)]^2/32}\, dx$$
$$= e^{8t^2-7t} \cdot 1 = e^{8t^2-7t}\,.$$

5. By the definition of cdf,

$$F_X(x) = P(X \le x) = P[F^{-1}(Y) \le x]\,.$$

If $F(x)$ is a valid cdf for a continuous random variable, it is strictly increasing. Thus, the condition $F^{-1}(Y) \le x$ is equivalent to the condition $Y \le F(x)$, and we get

$$F_X(x) = P[Y \le F(x)]\,.$$

Since Y is uniform over $(0,1)$, $F_Y(y) = y$, when $y \in [0,1]$. On the other hand, $F(x) \in [0,1]$ for all values of x, by definition. Hence,

$$F_X(x) = F_Y(F(x)) = F(x)\,.$$

This result means that random numbers from the uniform distribution over $(0,1)$ can be transformed into random numbers from the distribution defined by a cdf $F_X(x)$ by using the transformation $x = F_X^{-1}(y)$.

The cdf of the Cauchy distribution with parameter 1 is given by

$$F_X(x) = \int_{-\infty}^{x} \frac{dt}{\pi(1+t^2)} = \cdots = \frac{1}{\pi} \arctan x + \frac{1}{2}\,.$$

The required transformation is obtained by solving the following equation:

$$\begin{aligned}
&\frac{1}{\pi} \arctan x + \frac{1}{2} = y \\
\Rightarrow\ &\arctan x = \pi(y - \frac{1}{2}) \\
\Rightarrow\ &x = \tan\left[\,\pi(y - \frac{1}{2})\,\right] = F_X^{-1}(y)\,.
\end{aligned}$$