S-72.340 INFORMATION THEORY

Exercise 3, October 4, 2000

1. A random variable X can have non-negative integer values only. Find the probability assignment (or distribution) $p_X(k)$, $k = 0, 1, 2, \ldots$ that maximizes $H(X)$ subject to the constraint that the mean value of X,

$$\sum_{k=0}^{\infty} k\, p_X(k)$$

is a fixed value A. Evaluate the resulting $H(X)$.

2. Let X and Y denote two discrete random variables with joint distribution $P(x, y)$.

   a) Show that

$$H(X) = -\sum_{x,y} P(x, y) \log P(x)$$
$$H(Y) = -\sum_{x,y} P(x, y) \log P(y)$$

   b) Use the above result to show that $H(X, Y) \le H(X) + H(Y)$. When does equality hold?

   c) Show that $H(X|Y) \le H(X)$ with equality if and only if X and Y are independent.

3. Two binary random variables X and Y have the following joint distribution:

$$P(X = Y = 0) = P(X = 0, Y = 1) = P(X = Y = 1) = \frac{1}{3}.$$

   Compute $H(X)$, $H(Y)$, $H(X|Y)$, $H(Y|X)$ and $H(X, Y)$.

4. Let X be a continuous random variable with pdf $f_X(x)$, and let $Y = aX + b$ be a linear transformation of X, where a and b are two constants. Determine the (differential) entropy $H(Y)$ in terms of $H(X)$.

5. Let X_1 and X_2 be discrete random variables drawn according to probability mass functions (distributions) $p_1(x_1)$ and $p_2(x_2)$ over the respective alphabets $\mathbf{H_1} = \{1, 2, \ldots, m\}$ and $\mathbf{H_2} = \{m+1, \ldots, n\}$. Let

$$X = \begin{cases} X_1 & \text{with probability } \alpha \\ X_2 & \text{with probability } 1 - \alpha \end{cases}$$

a) Find $H(X)$ in terms of $H(X_1)$, $H(X_2)$ and α.

b) Maximize over α to show that

$$2^{H(X)} \leq 2^{H(X_1)} + 2^{H(X_2)}$$

and interpret using the notion that $2^{H(X)}$ is the effective alphabet size.

Try to solve as many problems as you can. Return your solutions by the time that the exercise begins. You can bring them to the exercise or put them into the 'return box' below the information board of our laboratory on the 2nd floor.