S-72.340 INFORMATION THEORY

Exercise 8, November 8, 2000 **Solutions**

1. We use again Theorem 4.4.1. from the lectures. As before, $H(\epsilon)$ denotes the binary entropy function.

$$
\begin{aligned}
I(x=0;Y) &= \sum_y P(y|0)\log\frac{P(y|0)}{P(y)} \\
&= (1-\epsilon)\log\frac{1-\epsilon}{p_0(1-\epsilon)} + \epsilon\log\frac{\epsilon}{p_0\epsilon+p_1} \\
&= -H(\epsilon) - (1-\epsilon)\log\left[p_0(1-\epsilon)\right] - \epsilon\log\left[1-p_0(1-\epsilon)\right]. \\
I(x=1;Y) &= \sum_y P(y|1)\log\frac{P(y|1)}{P(y)} \\
&= \log\frac{1}{p_0\epsilon+p_1} = -\log\left[1-p_0(1-\epsilon)\right]. \\
I(x=0;Y) &= I(x=1;Y) \\
\Rightarrow (1-\epsilon)\log\frac{1-p_0(1-\epsilon)}{p_0(1-\epsilon)} &= H(\epsilon) \\
\Rightarrow \frac{1-p_0(1-\epsilon)}{p_0(1-\epsilon)} &= 2^{\frac{H(\epsilon)}{1-\epsilon}} \\
\Rightarrow p_0 &= \frac{1}{(1-\epsilon)\left(1+2^{\frac{H(\epsilon)}{1-\epsilon}}\right)}.
\end{aligned}
$$

$$
\begin{aligned}
C &= -\log\left[1-p_0(1-\epsilon)\right] = -\log\frac{2^{\frac{H(\epsilon)}{1-\epsilon}}}{1+2^{\frac{H(\epsilon)}{1-\epsilon}}} \\
&= \log\left[1+2^{\frac{H(\epsilon)}{1-\epsilon}}\right] - \frac{H(\epsilon)}{1-\epsilon}.
\end{aligned}
$$

2. The output symbol y_n can be either 0 or 1:

$$
\begin{aligned}
g_n(s) &= \sum_{y_n} P(y_n|0)^{1-s}P(y_n|1)^s \\
&= P(0|0)^{1-s}P(0|1)^s + P(1|0)^{1-s}P(1|1)^s \\
&= (1-\epsilon)^{1-s}\cdot 0 + \epsilon\cdot 1^s = \epsilon = \text{ constant.} \\
P_{e,m} &\leq \left[\min g_n(s)\right]^N = \epsilon^N,\ m=1,2.
\end{aligned}
$$

3. In this case, y_n can be 0, E or 1:

$$\begin{aligned} g_n(s) &= \sum_{y_n} P(y_n|0)^{1-s} P(y_n|1)^s \\ &= P(0|0)^{1-s}P(0|1)^s + P(E|0)^{1-s}P(E|1)^s + P(1|0)^{1-s}P(1|1)^s \\ &= 0 + \epsilon^{1-s}\epsilon^s + 0 = \epsilon = \text{ constant.} \end{aligned}$$

So, in this case also, the upper bound becomes:

$$P_{e,m} \leq [\min g_n(s)]^N = \epsilon^N \,, \; m = 1, 2.$$

4. Maximum likelihood decoding of two codewords is defined here as follows: if an N-bit sequence $\mathbf{y}$ is received, codeword $\mathbf{x_1}$ is decoded if

$$P_N(\mathbf{y}|\mathbf{x_1}) > P_N(\mathbf{y}|\mathbf{x_2})\,;$$

otherwise $\mathbf{x_2}$ is decoded. In the case of a Z channel and these two codewords, this means that only if a sequence of N 1's is received, $\mathbf{x_2}$ is decoded; otherwise, the decoding result is $\mathbf{x_1}$.

Clearly, $P_{e,2} = 0$, but

$$P_{e,1} = \epsilon^N = 10^{-25} \,,$$

which is equal to the upper bound obtained in Problem 2.

5. The case of a binary erasure channel is also very simple. The only possibility of a decoding error is when a sequence of N erasure (E) symbols are received; in that case, $\mathbf{x_2}$ is decoded. Again, $P_{e,2} = 0$, and

$$P_{e,1} = \epsilon^N = 10^{-25} \,,$$

which is equal to the upper bound.