S-72.340 INFORMATION THEORY

Exercise 9, November 15, 2000 Solutions

1. We have $\mathbf{x_1} = 000\ldots0$ and $\mathbf{x_2} = 111\ldots1$. As in Exercise 8, we use maximum likelihood decoding, where $\mathbf{x_1}$ is chosen if $P(\mathbf{y}|\mathbf{x_1}) > P_N(\mathbf{y}|\mathbf{x_2})$, and $\mathbf{x_2}$ otherwise. The decoding rule divides the set of possible received vectors into two mutually exclusive parts, Y_1 and Y_2.

   Assume that $\mathbf{x_1}$ is transmitted. A decoding error occurs now if the received vector $\mathbf{y}$ is in the set $Y_1^c = Y_2$. This happens when the received vector contains $\lceil N/2 \rceil$ or more 1's, where $\lceil x \rceil$ is the smallest integer that is greater than or equal to x. The probability of decoding error is thus

$$P_{e,1} = \sum_{k=\lceil N/2 \rceil}^{N} \binom{N}{k} \epsilon^k (1-\epsilon)^{N-k}.$$

   If $\mathbf{x_2}$ is transmitted, a decoding error occurs if the received vector $\mathbf{y}$ contains $\lfloor N/2 \rfloor + 1$ or more 0's, where $\lfloor x \rfloor$ is the largest integer that is less than or equal to x. The probability of this is

$$P_{e,2} = \sum_{k=\lfloor N/2 \rfloor + 1}^{N} \binom{N}{k} \epsilon^k (1-\epsilon)^{N-k}.$$

   If $N = 25$ and $\epsilon = 0.1$, we get (with MATLAB):

$$P_{e,1} = P_{e,2} = \sum_{k=13}^{25} \binom{25}{k} 0.1^k \cdot 0.9^{25-k} \approx 1.6208 \cdot 10^{-7}.$$

   For comparison, the upper bound is given by Equation 5.1.17 from the lectures:

$$P_{e,m} \le \left[2 \cdot \sqrt{\epsilon \cdot (1-\epsilon)} \right]^N \approx 2.8430 \cdot 10^{-6}.$$

2. *Note:* There is an error in my solution to Problem 2 of Exercise 8. The function $g_n(s)$ is not a constant. It is given by

$$g_n(s) = \epsilon^{1-s} .$$

However, the minimum value of $g_n(s)$ in the interval $[0, 1]$ is still ϵ ; so, the error doesn't affect the final result.

In this problem, the codewords $\mathbf{x_1}$ and $\mathbf{x_2}$ are different from those in Exercise 8, and the function $g_n(s)$ is no longer independent of n . If $1 \leq n \leq N/2$,

$$\begin{aligned} g_n(s) &= \sum_{y_n} P(y_n|x_{1,n})^{1-s} P(y_n|x_{2,n})^s \\ &= P(0|0)^{1-s} P(0|1)^s + P(1|0)^{1-s} P(1|1)^s = \epsilon^{1-s} . \end{aligned}$$

If $N/2 < n \leq N$,

$$g_n(s) = P(0|1)^{1-s} P(0|0)^s + P(1|1)^{1-s} P(1|0)^s = \epsilon^s .$$

The upper bound becomes:

$$P_{e,m} \leq \prod_{n=1}^{N} g_n(s) = \epsilon^{N(1-s)/2} \cdot \epsilon^{Ns/2} = \epsilon^{N/2} .$$

The exact evaluation of $P_{e,1}$ and $P_{e,2}$ is very simple. If there are one or more 0's in the received vector $\mathbf{y}$, decoding is always correct. If $\mathbf{y}$ is a sequence of N 1's, the output of the decoder is $\mathbf{x_2}$. It is clear that $P_{e,2} = 0$, and $P_{e,1}$ is equal to the probability that $\mathbf{y}$ is a sequence of N 1's if $\mathbf{x_1}$ was transmitted:

$$P_{e,1} = \epsilon^{N/2} .$$

3. Assume that $\mathbf{x_1} = (x_{1,1}, \ldots , x_{1,N})$ is transmitted and $\mathbf{y} = (y_1, \ldots , y_N)$ is received. For each y_i , there are two possible input symbols $x_{1,i}$. Thus, the number of channel input sequences that can lead to $\mathbf{y}$ is 2^N . For each of these sequences,

$$P(\mathbf{y}|\mathbf{x_1}) = 2^{-N} .$$

We assume that if $P(\mathbf{y}|\mathbf{x_1}) = P(\mathbf{y}|\mathbf{x_2})$, the decoder always makes the wrong decision.

So, a decoding error occurs if and only if $\mathbf{x_2}$ is one of the 2^N input sequences that can lead to $\mathbf{y}$. The total number of possible channel input sequences is 3^N. All these sequences have the same probability 3^{-N}. Thus, the probability of decoding error is given by

$$P_e = \frac{2^N}{3^N} = \left(\frac{2}{3}\right)^N .$$

4. We assume again that ambiguity results in a decoding error. If either $\mathbf{x_1}$ or $\mathbf{x_2}$ is transmitted, a decoding error occurs if and only if the received vector $\mathbf{y}$ is a sequence of N 1's. The probability of this is

$$P_e = \left(\frac{1}{2}\right)^N .$$

5. If X is a Gaussian random variable with mean 0 and variance 1, then $Q(x) = P(X \geq x)$.

   It can be shown (see Proakis: *Digital Communications,* 3rd Ed., p. 55) that the Chernoff bound for $P(X \geq x)$ is given by

$$P(X \geq x) \leq e^{-\hat{s}x} E\left(e^{\hat{s}X}\right),$$

   where $\hat{s}$ is the solution to the equation

$$E\left(Xe^{sX}\right) = xE\left(e^{sX}\right).$$

$$\begin{aligned}
E\left(e^{sX}\right) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{st} e^{-t^2/2}\, dt \\
&= e^{s^2/2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(t-s)^2/2}\, dt = e^{s^2/2} \\
E\left(Xe^{sX}\right) &= \frac{d}{ds} E\left(e^{sX}\right) = se^{s^2/2}
\end{aligned}$$

So, we get

$$se^{s^2/2} = xe^{s^2/2} \;\Rightarrow\; \hat{s} = x \;\Rightarrow\; Q(x) \leq e^{-x^2} \cdot e^{x^2/2} = e^{-x^2/2}\,.$$