Exercises 26.01.2005

Exercise 1

Each time the experiment is conducted a certain number of Poisson distributed events occur. We know that for one hypothesis the mean value of events is m_0 and for the other m_1. The distributions follow Poisson distribution on both hypotheses.

Our observation is just number of samples and it ranges from 0 to ∞. Find the loglikelihood ratio?

Solution 1

The Poisson distribution is characterized as

$$\Pr\left(n \text{ events}\right) = \frac{\left(m_i\right)^2}{n!} e^{-m_i},\ n = 0, 1, \ldots,\ i = 0, 1.$$

Where the m_i is the parameter which specifies the average number of events: $E(n) = m_i$.

The parameter m_i is different in the two hypothesis. We have the two hypothesis:

$$H_1 : p_r\left(n \text{ events}\right) = \frac{m_1^n}{n!} e^{-m_1}$$

$$n = 0, 1, 2, \ldots.$$

$$H_0 : p_r\left(n \text{ events}\right) = \frac{m_0^n}{n!} e^{-m_0}$$

The likelihood ratio test is:

$$\Lambda(n) = \frac{m_1^n \big/ n!\, e^{-m_1}}{m_0^n \big/ n!\, e^{-m_0}} = \frac{m_1^n}{m_0^n} \exp\left(-\left(m_1 - m_0\right)\right) \underset{H_0}{\overset{H_1}{\gtrless}} \ln \eta$$

or equivalently,

$$n \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\ln \eta + m_1 - m_0}{\ln m_1 - \ln m_0} \quad \text{if } m_1 > m_0,$$

$$n \underset{H_1}{\overset{H_0}{\gtrless}} \frac{\ln \eta + m_1 - m_0}{\ln m_1 - \ln m_0} \quad \text{if } m_0 > m_1.$$

Exercise 2

A sample function of a simple Poisson counting process $N(t)$ is observed over the interval T:

Hypothesis H_1 : the mean rate is $k_1 : \Pr(H_1) = \frac{1}{2}$,

Hypothesis H_0 : the mean rate is $k_0 : \Pr(H_0) = \frac{1}{2}$.

1. Prove that the number of events in the interval T is a "sufficient statistic" to choose hypothesis H_0 or H_1.
2. Assuming equal costs for the possible errors, derive the appropriate likelihood ratio test and the threshold.

Solution 2

Let R denote the number of events over the interval T.

$$P_{r|H_1}(R|H_1) = \frac{(k_1 T)^R}{R!} e^{-k_1 T}$$
$$P_{r|H_0}(R|H_0) = \frac{(k_0 T)^R}{R!} e^{-k_0 T}$$
$$\Rightarrow \Lambda(R) = \frac{P_{r|H_0}(R|H_0)}{P_{r|H_1}(R|H_1)} = \left(\frac{k_1}{k_0}\right)^R e^{-(k_1-k_0)T}$$

$l(R) = \log(\Lambda(R)) = R + C$ is trivially a sufficient statistics since it is the observation itself.

Exercise 3

We observe the i.i.d. samples r for $n = 0, 1, \ldots, N-1$ from the Rayleigh pdf.

$$p_{r_i|H_k}(R_i \mid H_k) = \frac{R_i}{\sigma_k^2} \exp\left(-\frac{1}{2}\frac{R_i^2}{\sigma_k^2}\right).$$

Derive the Neyman-Pearson test for the hypothesis-testing problem:

$$\begin{aligned} &H_0 : \sigma_0^2 \\ &H_1 : \sigma_1^2 > \sigma_0^2 \end{aligned}.$$

Solution 3

Decide H_1 if

$$\frac{\prod_{n=0}^{N-1} \frac{r_n}{\sigma_1^2} e^{-\frac{1}{2}\frac{r_n^2}{\sigma_1^2}}}{\prod_{n=0}^{N-1} \frac{r_n}{\sigma_0^2} e^{-\frac{1}{2}\frac{r_n^2}{\sigma_0^2}}} = \frac{\sigma_1^{2N}}{\sigma_0^{2N}} e^{-\frac{1}{2}\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)\sum_{n=0}^{N-1} r_n^2} > \eta$$

$$\ln\left(\frac{\sigma_1^{2N}}{\sigma_0^{2N}}\right)-\frac{1}{2}\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)\sum_{n=0}^{N-1} r_n^2 > \ln(\eta)$$

$$\sum_{n=0}^{N-1} r_n^2 < \frac{2\left(\sigma_0^2\sigma_1^2\right)}{\left(\sigma_1^2-\sigma_0^2\right)}\left(\ln\left(\frac{\sigma_1^{2N}}{\sigma_0^{2N}}\right)-\ln(\eta)\right)$$

$$P_F = 1-\int_{\frac{2\left(\sigma_0^2\sigma_1^2\right)}{\left(\sigma_1^2-\sigma_0^2\right)}\left(\ln\left(\frac{\sigma_1^{2N}}{\sigma_0^{2N}}\right)-\ln(\eta)\right)}^{\infty} l_r \, dl_r$$

where $l_r = \sum_{n=0}^{N-1} r_n^2$

Exercise 4

The probability density of $\mathbf{r}$ on the three hypotheses is

$$p_{r_1,r_2|H_k}\left(R_1,R_2 \mid H_k\right)=\left(2\pi\sigma_{0k}\sigma_{1k}\right)^{-1}\exp\left[-\frac{1}{2}\left(\frac{R_1^2}{\sigma_{0k}^2}+\frac{R_2^2}{\sigma_{1k}^2}\right)\right],$$

where $-\infty < R_1, R_2 < \infty,\ k = 1,2,3$.

$\sigma_{00}^2=\sigma_{10}^2=\sigma_n^2,$

$\sigma_{01}^2=\sigma_s^2+\sigma_n^2,\ \sigma_{11}^2=\sigma_n^2,$

$\sigma_{02}^2=\sigma_n^2, \qquad \sigma_{12}^2=\sigma_s^2+\sigma_n^2.$

The cost matrix is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & \alpha \\ 1 & \alpha & 0 \end{bmatrix},$$

where $0 \le \alpha < 1$ and $\Pr\left(H_1\right)=\Pr\left(H_2\right)\triangleq p$. Define $l_1 = R_1^2$ and $l_2 = R_2^2$.

1. Find the optimum test and indicate the decision regions in the l_1, l_2-plane.
2. Write an expression for the error probabilities. (Do not evaluate the integrals.)

Solution 4

Let $\Lambda_1(R)$ denote the likelihood ratio between the second and the first hypothesis and $\Lambda_2(R)$ between the third and first.

$$\Lambda_1(R)=\frac{P\left(R|H_1\right)}{P\left(R|H_0\right)}=\frac{\sigma_n}{\sqrt{\sigma_s^2+\sigma_n^2}}\exp\left[\frac{R_1^2}{2}\frac{\sigma_s^2}{\sigma_n^2\left(\sigma_s^2+\sigma_n^2\right)}\right]$$

$$\Lambda_2(R)=\frac{P\left(R|H_2\right)}{P\left(R|H_0\right)}=\frac{\sigma_n}{\sqrt{\sigma_s^2+\sigma_n^2}}\exp\left[\frac{R_2^2}{2}\frac{\sigma_s^2}{\sigma_n^2\left(\sigma_s^2+\sigma_n^2\right)}\right]$$

Optimum test becomes

$$P_1\left(C_{01}-C_{11}\right)\Lambda_1(R)\underset{H_0 \text{ or } H_2}{\overset{H_1 \text{ or } H_2}{\lessgtr}}P_0\left(C_{10}-C_{00}\right)+P_2\left(C_{12}-C_{02}\right)\Lambda_2(R)$$

$$P_2\left(C_{02}-C_{22}\right)\Lambda_2(R)\underset{H_0 \text{ or } H_1}{\overset{H_1 \text{ or } H_2}{\lessgtr}}P_0\left(C_{20}-C_{00}\right)+P_1\left(C_{21}-C_{01}\right)\Lambda_1(R)$$

$$P_2\left(C_{12}-C_{22}\right)\Lambda_2(R)\underset{H_0 \text{ or } H_1}{\overset{H_0 \text{ or } H_2}{\lessgtr}}P_0\left(C_{20}-C_{10}\right)+P_1\left(C_{21}-C_{11}\right)\Lambda_1(R)$$

By inserting into the equations $\Pr(H_1) = \Pr(H_2) \triangleq p$ and $\Pr(H_0) \triangleq 1-2p$

$$p\Lambda_1(R) \underset{H_0 \text{ or } H_2}{\overset{H_1 \text{ or } H_2}{\gtrless}} (1-1p) + p(\alpha-1)\Lambda_2(R)$$

$$p\Lambda_2(R) \underset{H_0 \text{ or } H_1}{\overset{H_1 \text{ or } H_2}{\gtrless}} (1-2p)$$

$$p\alpha\Lambda_2(R) \underset{H_0 \text{ or } H_1}{\overset{H_0 \text{ or } H_2}{\gtrless}} p\Lambda_1(R)$$

The second equation

$$p\Lambda_2(R) \underset{H_0 \text{ or } H_1}{\overset{H_1 \text{ or } H_2}{\gtrless}} (1-2p)$$

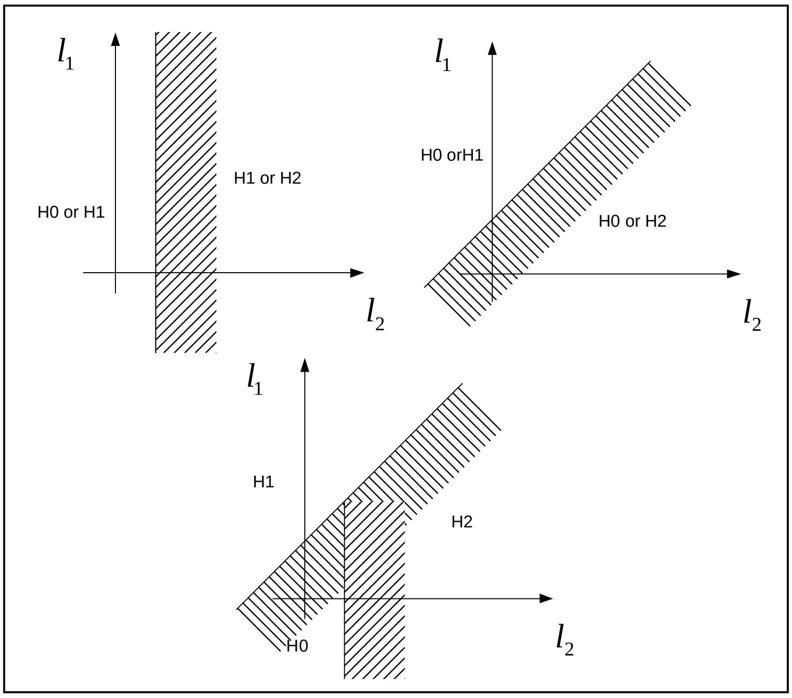
$$\log(p) + \log(\Lambda_2(R)) \underset{H_0 \text{ or } H_1}{\overset{H_1 \text{ or } H_2}{\gtrless}} \log(1-2p)$$

$$l_2 \underset{H_0 \text{ or } H_1}{\overset{H_1 \text{ or } H_2}{\gtrless}} \underbrace{\frac{2\sigma_n^2\left(\sigma_s^2+\sigma_n^2\right)}{\sigma_s^2}}_{>0}\left[\ln\left(\frac{1-2p}{p}\right) + \ln\left(\frac{\sqrt{\sigma_s^2+\sigma_n^2}}{\sigma_n}\right)\right]$$

First part always bigger than zero, second part also.The equation does not depend on the l_1 value.

The third equation

$$\ln\alpha + l_1\frac{\sigma_s^2}{2\sigma_n^2\left(\sigma_s^2+\sigma_n^2\right)} \underset{H_0 \text{ or } H_1}{\overset{H_1 \text{ or } H_2}{\gtrless}} l_2\frac{\sigma_s^2}{2\sigma_n^2\left(\sigma_s^2+\sigma_n^2\right)}$$



$$l_1 \underset{H_0 \text{ or } H_1}{\overset{H_1 \text{ or } H_2}{\lesseqgtr}} l_2 - \frac{2\sigma_n^2\left(\sigma_s^2 + \sigma_n^2\right)}{\sigma_s^2} \ln \alpha$$

2. Error probabilities

$$\Pr\left(H_1 H_2 \middle| H_0\right) = \int_{Z_0} \left[p P_{r_1 r_2 | H_2}\left(R_1, R_2 \middle| H_2\right) + p P_{r_1 r_2 | H_1}\left(R_1, R_2 \middle| H_1\right) \right] dR_1 dR_2$$

$$\Pr\left(H_0 H_2 \middle| H_1\right) = \int_{Z_1} \left[(1-2p) P_{r_1 r_2 | H_0}\left(R_1, R_2 \middle| H_0\right) + p\alpha P_{r_1 r_2 | H_2}\left(R_1, R_2 \middle| H_2\right) \right] dR_1 dR_2$$

$$\Pr\left(H_0 H_1 \middle| H_2\right) = \int_{Z_2} \left[(1-2p) P_{r_1 r_2 | H_0}\left(R_1, R_2 \middle| H_0\right) + p\alpha P_{r_1 r_2 | H_1}\left(R_1, R_2 \middle| H_1\right) \right] dR_1 dR_2$$

Where Z_i are the decision regions associated with the hypothesis H_i.