Tutorial 5

Exercise 1

DC level in AWGN with unknown Amplitude.

We have N samples with unknown amplitude A. The channel is known to be AWGN channel with noise variance σ^2.

We have a hypothesis testing problem: $\begin{array}{l} H_1 : A \neq 0, \\ H_0 : A = 0. \end{array}$

Where in case of H_1 A is an unknown constant.

Find the General Likelihood Ratio Test.

Solution 1

Recall that the GLRT is

$$\Lambda_g(\mathrm{R}) = \frac{\max\limits_{\theta_1} p_{\mathrm{r}|\theta_1}(\mathrm{R} \mid \theta_1)}{\max\limits_{\theta_0} p_{\mathrm{r}|\theta_0}(\mathrm{R} \mid \theta_0)} \underset{H_1}{\overset{H_0}{\lessgtr}} \gamma$$

The likelihood ratio test is:

$$\Lambda(R) = \frac{p\left(R; \hat{A}, H_1\right)}{p\left(R; H_0\right)} > \eta \ .$$

The Maximum Likelihood Estimation for $\hat{A}$ is found by maximizing:

$$p\left(R; \hat{A}, H_1\right) = \frac{1}{\left(2\pi\sigma^2\right)^{\frac{N}{2}}} \exp\left[\frac{1}{2\sigma^2} \sum_{i=0}^{N-1} \left(R_i - A\right)^2\right].$$

By maximizing this we can express the estimation of A as the mean $R_i \Rightarrow \hat{A} = \overline{R} = \frac{1}{N} \sum_{i=0}^{N-1} R_i$.

$$\Lambda(R) = \frac{p\left(R;\hat{A},H_1\right)}{p\left(R;H_0\right)} = \frac{\frac{1}{\left(2\pi\sigma^2\right)^{\frac{N}{2}}}\exp\left[\frac{1}{2\sigma^2}\sum_{i=0}^{N-1}\left(R_i-\overline{R}\right)^2\right]}{\frac{1}{\left(2\pi\sigma^2\right)^{\frac{N}{2}}}\exp\left[\frac{1}{2\sigma^2}\sum_{i=0}^{N-1}\left(R_i\right)^2\right]}.$$

Taking the logarithm we have

$$-\frac{1}{2\sigma^2}\left(\sum_{i=0}^{N-1}R_i^2 - 2\overline{R}\sum_{i=0}^{N-1}R_i + N\overline{R} - \sum_{i=0}^{N-1}R_i^2\right)$$

$$= \frac{1}{2\sigma^2}\left(-2N\overline{R}^2 + N\overline{R}^2\right) = \frac{N\overline{R}^2}{2\sigma^2}$$

We decide H_1 if: $\overline{R}^2 > \gamma'$.

Exercise 2

The observation consists of a set of values of the random variables, $r_1, r_2, \ldots, r_m$

$r_i = s_i + n_i,\ i = 0,1,2,\ldots,N-1,\ H_1$

$r_i = n_i\ \ i = 0,1,2,\ldots,N-1\ \ H_0$

The s_i and n_i are independent, identically distributed random variables with densities $N(0,\sigma_s)$ and $N(0,\sigma_n)$ respectively, where σ_n is known and σ_s is unknown.

1. Does the UMP test exist?

2. If the answer to part 1 is negative, find the generalized LRT.

Solution 2

We can find a posteriori distributions for both hypotheses:

$$P_{r|H_0}\left(R|\,H_0\right) = \frac{1}{\left(2\pi\sigma_n^2\right)^{\frac{N-1}{2}}}\,\mathrm{e}^{-\frac{\sum_{i=0}^{N-1}R_i^2}{2\sigma_n^2}}$$

$$P_{r|H_1}\left(R|\,H_1\right) = \frac{1}{\left(2\pi\left(\sigma_n^2+\sigma_s^2\right)\right)^{\frac{N-1}{2}}}\,\mathrm{e}^{-\frac{\sum_{i=0}^{N-1}R_i^2}{2\left(\sigma_n^2+\sigma_s^2\right)}}$$

The test is simple division of these distributions

$$\Lambda(R)=\frac{P\left(R\mid H_1\right)}{P\left(R\mid H_0\right)}=\left[\frac{\sigma_n^2}{\sigma_s^2+\sigma_n^2}\right]^{\frac{N-1}{2}}\exp\left[-\frac{\sum_{i=0}^{N-1}R_i^2}{2\left(\sigma_n^2+\sigma_s^2\right)}+\frac{\sum_{i=0}^{N-1}R_i^2}{2\sigma_n^2}\right]\lessgtr\eta$$

$$\ln\left(\Lambda(R)\right)=\frac{N-1}{2}\ln\left[\frac{\sigma_n^2}{\sigma_s^2+\sigma_n^2}\right]+\left[\frac{\left(\sigma_n^2+\sigma_s^2\right)-\sigma_n^2}{2\sigma_n^2\left(\sigma_n^2+\sigma_s^2\right)}\sum_{i=0}^{N-1}R_i^2\right]\lessgtr\ln(\eta)$$

$$\sum_{i=0}^{N-1}R_i^2\lessgtr\left[2\ln(\eta)+(N-1)\ln\left[1+\frac{\sigma_s^2}{\sigma_n^2}\right]\right]\left(\sigma_n^2+\frac{\sigma_n^4}{\sigma_s^2}\right)$$

The LRT test can not be specified without knowing σ_s^2.

UMT test does not exist.

Generalized LRT

ML estimate of $\Theta=\sigma_s^2$ when H_1 is true

$$\ln\left(P_{r|H_1}\left(R\mid\Theta\right)\right)=-\frac{N-1}{2}\left[\ln(2\pi)+\ln\left(\sigma_n^2+\Theta\right)\right]-\frac{\sum_{i=0}^{N-1}R_i^2}{2\left(\sigma_n^2+\Theta\right)}$$

$$\frac{\partial}{\partial\Theta}\ln\left(P_{r|H_1}\left(R\mid\Theta\right)\right)=-\frac{N-1}{2\left(\sigma_n^2+\Theta\right)}+\frac{\sum_{i=0}^{N-1}R_i^2}{2\left(\sigma_n^2+\Theta\right)^2}=0$$

$$(N-1)\left(\sigma_n^2+\Theta\right)=\sum_{i=0}^{N-1}R_i^2\Rightarrow\hat{\Theta}_{ML}=\frac{1}{N-1}\sum_{i=0}^{N-1}R_i^2-\sigma_n^2$$

Substituting this for σ_s^2 in the LRT in part 1 and denoting

$$l(R)=\sum_{i=0}^{N-1}R_i^2$$

$$l(R)\lessgtr\left[2\ln(\eta)+(N-1)\ln\left[1+\frac{\frac{1}{N-1}l(R)-\sigma_n^2}{\sigma_n^2}\right]\right]\left(\sigma_n^2+\frac{\sigma_n^4}{\frac{1}{N-1}l(R)-\sigma_n^2}\right)$$

$$l(R)\lessgtr\left[2\ln(\eta)+(N-1)\ln\left[\frac{N-2}{N-1}+\frac{1}{(N-1)\sigma_n^2}l(R)\right]\right]\sigma_n^2\left(\frac{l(R)}{l(R)-(N-1)\sigma_n^2}\right)$$

$$\frac{\left(l(R)-(N-1)\sigma_n^2\right)}{\sigma_n^2}-(N-1)\ln\left[\frac{N-2}{N-1}+\frac{1}{(N-1)\sigma_n^2}l(R)\right]\lessgtr2\ln(\eta)$$

$$\frac{l(R)}{\sigma_n^2}-(N-1)\ln\left[\frac{(N-2)\sigma_n^2+l(R)}{(N-1)\sigma_n^2}\right]\lessgtr2\ln(\eta)-(N-1)$$

$$\frac{l(R)}{\sigma_n^2} - (N-1)\ln\left((N-2)\sigma_n^2 + l(R)\right) \lessgtr 2\ln(\eta) - (N-1)\left(1 + \ln\left((N-1)\sigma_n^2\right)\right)$$

Exercise 3

Consider the detection of a signal s_i embedded in AWGN with variance σ^2 based on the observed samples r_i for $i = 0, 1, \dots, 2N-1$. The signal is given by

$$H_0 = \begin{cases} A & n = 0, 1, \dots, N-1 \\ 0 & n = N, N+1, \dots, 2N-1 \end{cases}$$

$$H_1 = \begin{cases} A & n = 0, 1, \dots, N-1 \\ 2A & n = N, N+1, \dots, 2N-1 \end{cases}$$

Assume that $A > 0$ and find the NP detector as well as its detection performance.

Solution 3

$$\Lambda(R) = \frac{p(R|H_1)}{p(R|H_0)} > \eta$$

$$P(R|H_0) = \frac{1}{(2\pi\sigma^2)^N} e^{-\frac{1}{2\sigma^2}\sum_{i=0}^{N-1}(R_i - A)^2} e^{-\frac{1}{2\sigma^2}\sum_{i=N}^{2N-1}(R_i - 0)^2}$$

$$P(R|H_0) = \frac{1}{(2\pi\sigma^2)^N} e^{-\frac{1}{2\sigma^2}\sum_{i=0}^{N-1}(R_i - A)^2} e^{-\frac{1}{2\sigma^2}\sum_{i=N}^{2N-1}(R_i - 2A)^2}$$

log-likelihood ratio

$$\ln(\Lambda(R)) = \frac{1}{2\sigma^2}\sum_{i=N}^{2N-1}(R_i - 0)^2 - \frac{1}{2\sigma^2}\sum_{i=N}^{2N-1}(R_i - 2A)^2$$

$$\ln(\Lambda(R)) = \frac{1}{2\sigma^2}\sum_{i=N}^{2N-1}(4AR_i - 4NA^2) = \frac{4A}{2\sigma^2}\sum_{i=N}^{2N-1}R_i + \frac{4NA^2}{2\sigma^2} > \ln(\eta)$$

We can multiply with $\frac{\sigma^2}{2A}$ and subtract $4NA^2$. That changes the comparison level but does not change the test.

$$\sum_{i=N}^{2N-1} R_i > \frac{\sigma^2}{2A}\left(\ln(\eta) - \frac{2\sigma^2}{4NA^2}\right)$$

$$\frac{1}{N}\sum_{i=N}^{2N-1} R_i > \frac{\sigma^2}{2NA}\ln(\eta) - A$$

The test is to take the mean of the last N samples.

$$l(R) = \frac{1}{N}\sum_{n=0}^{N-1} R_i$$

The test statistics is Gaussian for both hypothesis

$$H_0: \quad E\{l(R)\} = 0$$

$$Var\{l(R)\} = \frac{\sigma^2}{N}$$

$$H_1: \quad E\{l(R)\} = 2A$$

$$Var\{l(R)\} = \frac{\sigma^2}{N}$$

$$P_F : P\{l(R) > \eta' | H_0\} = Q\left(\frac{\eta'}{\sqrt{\sigma^2 / N}}\right)$$

$$P_D : P\{l(R) > \eta' | H_1\} = Q\left(\frac{\eta' - 2A}{\sqrt{\sigma^2 / N}}\right)$$