Tutorial 4.

Exercise 1

Three input binary bits (with levels $u \in \{-1,1\}$) are mixed together accordingly to the matrix

$$\mathbf{H} = \begin{bmatrix} -0.97 & -0.09 & -0.29 \\ -1.51 & -1.68 & -1.13 \\ 1.41 & 1.14 & 0.4 \end{bmatrix}$$

The resulting signal is

$$\mathbf{y} = \begin{bmatrix} 0.747 \\ 2.49 \\ -2.40 \end{bmatrix}$$

Search for the most likely bit sequence by using sphere decoder with the radius $C = 1$.

Solution 1

The problem to be solved can be formulated as

$$\hat{\mathbf{x}} = \arg\min \left\{ \left| \mathbf{y} - \mathbf{H}\mathbf{x} \right|^2 \right\}$$

where $\mathbf{x}$ describes the vectors of any possible bit combination. For 3 bits we have 8 possible bit sequences.

The mixing matrix can be modified by using qr factorization.

$$\mathbf{H} = \mathbf{QR} = \begin{bmatrix} -0.425 & 0.839 & 0.341 \\ -0.662 & -0.545 & 0.515 \\ 0.618 & 0.007 & 0.786 \end{bmatrix} \begin{bmatrix} 1.02 & 0.832 & 0.502 \\ 0 & 0.374 & 0.166 \\ 0 & 0 & -0.165 \end{bmatrix}$$

We are searching through all the bit sequences for which

$$\left| \mathbf{y} - \mathbf{H}\mathbf{x} \right|^2 < C$$

$$\left| \mathbf{Q}^H \mathbf{y} - \mathbf{Q}^H \mathbf{QRx} \right|^2 = \left| \mathbf{y}' - \mathbf{Rx} \right|^2 < C$$

$$\mathbf{Q}^H \mathbf{y} = \mathbf{y}' = \begin{bmatrix} -3.44 \\ -0.72 \\ -0.35 \end{bmatrix}$$

$$\left\| \begin{bmatrix} -3.44 \\ -0.72 \\ -0.35 \end{bmatrix} - \begin{bmatrix} 1.02 & 0.832 & 0.502 \\ 0 & 0.374 & 0.166 \\ 0 & 0 & -0.165 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\|^2 < C$$

That matrix form can be expressed also as the sum of the powers of the individual rows.

$$\left|-0.35-0.165x_3\right|^2+\left|-0.72-0.374x_2-0.166x_3\right|^2+\left|-3.44-1.02x_3-0.832x_2-0.502x_1\right|^2<C$$

We fix x_3. We see that if the sum containing only x_3 exceeds already C we do not have to look into possible x_2, x_1 i.e. to vectors where for fixes x_3 distance from received point can only increase. We can select new x_3 and check for it. Similar property exist only for x_2. By not looking into some vectors we limit out search space.

A possible flowchart of the algorithm is outlined in the reference [1] .

We start by initializing

$$\mathbf{x}=\begin{bmatrix}\times\\\times\\-1\end{bmatrix}$$

$$\left|-0.35-0.165x_3\right|^2=\left|-0.35-0.165\cdot(-1)\right|^2=0.516$$

It is less than C and we can proceed by fixing x_2.

$$\mathbf{x}=\begin{bmatrix}\times\\-1\\-1\end{bmatrix}$$

The cost for given x_2, x_3 is

$$\begin{aligned}&\left|-0.35-0.165x_3\right|^2+\left|-0.72-0.374x_2-0.166x_3\right|^2\\&=\left|-0.35-0.165\cdot(-1)\right|^2+\left|-0.72-0.374\cdot(-1)-0.166\cdot(-1)\right|^2\\&=0.748\end{aligned}$$

That also is less than C and we can proceed to x_1

$$\mathbf{x}=\begin{bmatrix}-1\\-1\\-1\end{bmatrix}$$

$$\begin{aligned}&\left|-0.35-0.165x_3\right|^2+\left|-0.72-0.374x_2-0.166x_3\right|^2+\left|-3.44-1.02x_3-0.832x_2-0.502x_1\right|^2\\&=\begin{array}{l}\left|-0.35-0.165\cdot(-1)\right|^2+\left|-0.72-0.374\cdot(-1)-0.166\cdot(-1)\right|^2\\\quad+\left|-3.44-1.02\cdot(-1)-0.832\cdot(-1)-0.502\cdot(-1)\right|^2\end{array}\\&=4.02\end{aligned}$$

Now we exceed our limit C and we can exclude this vector.

Since there is no lower level we select next possible value for x_1

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$d^2(\mathbf{y},\mathbf{x}) \geq 8.34$

Also this vector is excluded.

Since there is no other possible value for x_1 we move one layer up and assign next possible value to x_2.

$$\mathbf{x} = \begin{bmatrix} \times \\ 1 \\ -1 \end{bmatrix}$$

$$\left|-0.35-0.165x_3\right|^2+\left|-0.72-0.374x_2-0.166x_3\right|^2$$
$$=\left|-0.35-0.165\cdot(-1)\right|^2+\left|-0.72-0.374\cdot(1)-0.166\cdot(-1)\right|^2$$
$$=1.91$$

This vector is already exceeding C and vector that have at positions 2 and 3 those bits will exceed C.

We attempt to assign next possible value for x_2. Since there none we move up to the higher level – 3.

We can go to next branch

$$\mathbf{x} = \begin{bmatrix} \times \\ \times \\ 1 \end{bmatrix}$$

$$\left|-0.35-0.165x_3\right|^2=\left|-0.35-0.165\cdot(1)\right|^2=0.0002$$

$$\mathbf{x} = \begin{bmatrix} \times \\ -1 \\ 1 \end{bmatrix}$$

$d^2(\mathbf{y},\mathbf{x}) \geq 0.067$

$$\mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$d^2(\mathbf{y},\mathbf{x}) \geq 0.24$

That vector satisfies our requirement to belong into sphere and we will remember it.

We assign next possible value to the x_1

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$d^2(\mathbf{y}, \mathbf{x}) \geq 25$

That vector exceeds the allowed distance and we exclude it from the possible set of vectors.

At level 1 there is no more possible x values left we move up to level 2.

$$\mathbf{x} = \begin{bmatrix} \times \\ 1 \\ 1 \end{bmatrix}`$$

$d^2(\mathbf{y}, \mathbf{x}) \geq 3.69$

This vector is also excluded from the search.

We move up to the level 3. Since there is no possible bit value left we stop the search.

The only possible value for the vector we found is

$$\hat{\mathbf{x}} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Refrence.

[1] M.O. Damen, H.ElGamal, " On maximum-likelihood detection and the search for the closest lattice point", IEEE transactions on information theory, vol. 49, no. 10. October 2003, pp. 2389-2402

Exercise 2

Preparation of the threaded code for decoding with the sphere decoder.
We have a system with 2 transmit antenna and 2 receive antenna. In the system we have two threads separated by the algebraic number $\phi = i$.
Into one threads are mapped two binary symbols with possible values $u \in \{-1,1\}$ and rotation matrix $M = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$. In one block with duration 2 time moments are transmitted 4 bits of information. If we would use also the complex values for u total datarate would top 8 bits of information per block.
The channel matrix is

$$H = \begin{bmatrix} -0.58+0.37 & 0.32-0.37 \\ 0.051+0.56 & -0.27-0.55 \end{bmatrix},$$

where the rows describe the transmitters and columns corresponds to the receivers.

The threads are mapped as follows.
$Thread_1 = \{1,2\}$ meaning in the first interval we transmit the stream 1 from the antenna 1, in the second interval from the antenna 2.
$Thread_2 = \{2,1\}$ meaning in the first in interval we transmit the stream 2 from the antenna 2, in the second interval from the antenna 1.

The converted transition matrix as seen by the bits in the thread 1 is

$$H_1 = \begin{bmatrix} -0.58+0.37i & 0 \\ 0 & 0.051+0.56i \\ 0.32-0.37i & 0 \\ 0 & -0.27-0.55i \end{bmatrix}.$$

For the bits in the thread 2 it is

$$H_2 = \begin{bmatrix} 0.051+0.56i & 0 \\ 0 & -0.58+0.37i \\ -0.27-0.55i & 0 \\ 0 & 0.32-0.37i \end{bmatrix}.$$

The united channel matrix is

$$H = \begin{bmatrix} -0.58+0.37i & 0 & 0.051+0.56i & 0 \\ 0 & 0.051+0.56i & 0 & -0.58+0.37i \\ 0.32-0.37i & 0 & -0.27-0.55i & 0 \\ 0 & -0.27-0.55i & 0 & 0.32-0.37i \end{bmatrix}$$

The rotation matrix M that describes both the rotation for the bits in the thread one and two is

$$M = \begin{bmatrix} \begin{matrix} 1 & i \\ 1 & -i \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & i\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & i & 0 & 0 \\ 1 & -i & 0 & 0 \\ 0 & 0 & i & -1 \\ 0 & 0 & i & 1 \end{bmatrix}$$

In construction of the rotation matrix we assumed that the bits are ordered: two first in the row are bits in the first thread $\mathbf{u}_{th,1} = \begin{bmatrix} u_{t,1,bit1} \\ u_{t,1,bit2} \end{bmatrix}$ and the lower two are the bits in the thread 2 $\mathbf{u}_{th,2} = \begin{bmatrix} u_{t,2,bit1} \\ u_{t,2,bit2} \end{bmatrix}$.

$$u = \begin{bmatrix} \mathbf{u}_{th,1} \\ \mathbf{u}_{th,2} \end{bmatrix} = \begin{bmatrix} u_{t,1,bit1} \\ u_{t,1,bit2} \\ u_{t,2,bit1} \\ u_{t,2,bit2} \end{bmatrix}$$

By combing the channel matrix and the rotation matrix we get

$$\mathbf{B} = \mathbf{HM} = \begin{bmatrix} -0.58+0.37i & -0.37-0.58i & -0.56+0.051i & -0.051-0.56i \\ 0.051+0.56i & 0.56-0.051i & -0.37-0.58i & -0.58+0.37i \\ 0.32-0.37i & 0.37-0.32i & 0.55-0.27i & 0.27+0.55i \\ -0.27-0.55i & -0.55+0.27i & 0.37+0.32 & 0.32-0.37i \end{bmatrix}$$

The matrix can be converted to be a full real matrix by using

$$\mathbf{B}_{real} = \begin{bmatrix} \mathrm{Re}(\mathbf{B}) & -\mathrm{Im}(\mathbf{B}) \\ \mathrm{Im}(\mathbf{B}) & \mathrm{Re}(\mathbf{B}) \end{bmatrix}$$

$$\mathbf{B}_{real} = \begin{bmatrix} -0.58 & -0.37 & -0.56 & -0.051 & -0.37 & 0.58 & -0.51 & 0.56 \\ 0.051 & 0.56 & -0.37 & -0.58 & -0.56 & 0.051 & 0.58 & -0.37 \\ 0.32 & 0.37 & 0.55 & 0.27 & 0.37 & -0.32 & 0.27 & -0.55 \\ -0.27 & -0.55 & 0.37 & 0.37 & 0.55 & 0.55 & -0.32 & 0.37 \\ 0.37 & -0.58 & 0.051 & -0.56 & -0.58 & -0.58 & -0.56 & -0.051 \\ 0.56 & -0.051 & -0.58 & 0.37 & 0.051 & 0.051 & -0.37 & -0.58 \\ -0.37 & 0.32 & -0.27 & 0.55 & 0.32 & 0.32 & 0.55 & 0.27 \\ -0.55 & 0.27 & 0.32 & -0.37 & -0.27 & -0.27 & 0.37 & 0.32 \end{bmatrix}$$

The received vector $\mathbf{y}$ is also converted into real vector

$$\mathbf{y}_{real} = \begin{bmatrix} \mathrm{Re}(\mathbf{y}) \\ \mathrm{Im}(\mathbf{y}) \end{bmatrix}$$

For simplifying the search of the distance from the received point we split the matrix

$\mathbf{B}_{real}$ into 'qr' form: $\mathbf{QR} = \mathbf{B}_{real}$

$$\mathbf{Q} = \begin{bmatrix} -0.48 & -0.31 & 0.47 & 0.043 & 0.33 & 0.39 & -0.12 & -0.39 \\ 0.043 & 0.47 & 0.31 & 0.49 & -0.39 & -0.12 & -0.39 & -0.33 \\ 0.27 & 0.31 & -0.46 & -0.22 & 0.48 & 0.22 & -0.33 & -0.48 \\ -0.23 & -0.46 & -0.31 & -0.27 & -0.33 & -0.33 & -0.22 & -0.41 \\ 0.31 & -0.49 & -0.043 & 0.47 & 0.39 & 0.33 & -0.39 & 0.12 \\ 0.47 & -0.043 & 0.48 & -0.31 & 0.12 & -0.39 & 0.33 & -0.39 \\ -0.31 & -0.27 & 0.23 & -0.46 & 0.22 & -0.41 & -0.33 & 0.32 \\ -0.46 & 0.23 & -0.27 & 0.31 & 0.32 & -0.48 & 0.41 & -0.22 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 1.19 & 0 & 0 & 0 & 0 & 0.018 & -0.5 & -1.05 \\ 0 & -1.19 & 0 & 0 & -0.018 & 0 & -1.05 & 0.5 \\ 0 & 0 & -1.19 & 0 & -0.5 & 1.05 & 0 & -0.018 \\ 0 & 0 & 0 & -1.19 & -1.05 & -0.5 & -0.018 & 0 \\ 0 & 0 & 0 & 0 & -0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 \end{bmatrix}$$

The search has now a form

$$|\mathbf{y} - \mathbf{Hx}|^2 < C$$

$$|\mathbf{y} - \mathbf{HMu}|^2 < C$$

where $\mathbf{x}$ is calculated from $\mathbf{u}$: $\mathbf{x} = \mathbf{Mu}$ and we can modify the search into from

$$|\mathbf{Q}^H\mathbf{y} - \mathbf{Q}^H\mathbf{QRu}|^2 = |\mathbf{y}' - \mathbf{Ru}|^2 < C$$

Here the matrix $\mathbf{R}$ has size 8×8 but we are transmitting only 4 bits. We send less information than possible since we are not using the complex values for $\mathbf{u}$. The positions 5-8 are not used and value of $\mathbf{u}$ in those positions is 0.

$$\mathbf{u} = \begin{bmatrix} u_{t,1,bit1} \\ u_{t,1,bit2} \\ u_{t,2,bit1} \\ u_{t,2,bit2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In this particular case the search for the values of $\mathbf{u}$ can be reduced to search in four positions only.

Refrence.

[2] H.ElGamal, M.O.Damen, " Universal Space-Time Coding", IEEE transactions on information theory, Vol. 49, No. 5, May 2003.