S-72.1140 Transmission Methods in Telecommunication Systems (5 cr)

Exponential Carrier Wave Modulation

Exponential modulation: Frequency (FM) and phase (PM) modulation

- FM and PM waveforms
- Instantaneous frequency and phase
- Spectral properties
  - narrow band
    - arbitrary modulating waveform
    - tone modulation - phasor diagram
  - wideband tone modulation
  - Transmission BW
- Generating FM: de-tuned tank circuit
- Generating PM: narrow band mixer modulator
- Generating FM/PM: indirect modulators (more linear operation - What linearity means here?)

Contents (cont.)

- Detecting FM/PM
  - FM-AM conversion followed by envelope detector
  - Phase-shift discriminator
  - Zero-crossing detection (tutorials)
  - PLL-detector (tutorials)
- Effect of additive interference on FM and PM
  - analytical expressions and phasor diagrams
  - implications for demodulator design
- FM preemphases and deemphases filters

Linear and exponential modulation

- In linear CW (carrier wave) modulation:
  - transmitted spectra resembles modulating spectra
  - spectral width does not exceed twice the modulating spectral width
  - destination SNR can not be better than the baseband transmission SNR (lecture: Noise in CW systems)
- In exponential CW modulation:
  - usually transmission BW>>baseband BW
  - bandwidth-power trade-off (channel adaptation)
  - baseband and transmitted spectra does not carry a simple relationship

Phase modulation (PM)

- Carrier Wave (CW) signal: $x_c(t) = A_c \cos(\underbrace{\omega_c t + \phi(t)}_{\theta_c(t)})$
- In exponential modulation the modulation is "in the exponent" or "in the angle"

$$x_c(t) = A_c \cos(\theta_c(t)) = A_c \operatorname{Re}[\exp(j\theta_c(t))]$$

- Note that in exponential modulation superposition does not apply:

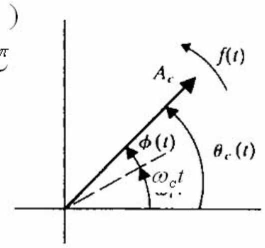
$$x_c(t) = A\cos\{\omega_c t + k_f[a_1(t) + a_2(t)]\}$$

$$\neq A\cos\omega_c t + A\cos k_f[a_1(t) + a_2(t)]$$

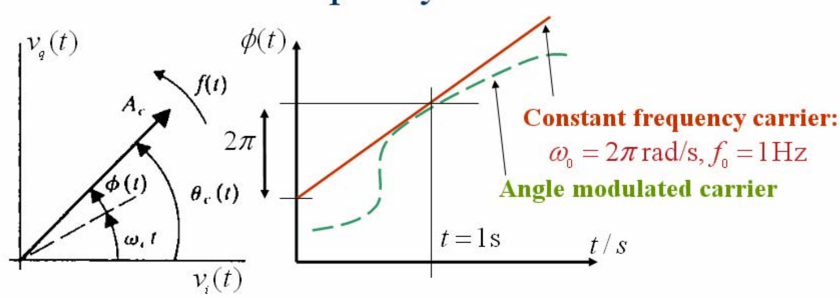
- In **phase modulation** (PM) carrier phase is linearly proportional to the modulation amplitude:

$$x_{PM}(t) = A_c \cos(\underbrace{\omega_c t + \underbrace{\phi(t)}_{\phi_\Delta x(t), \phi_\Delta \le \pi}}_{\theta_C(t)})$$

- Angular phasor has the *instantaneous* frequency (*phasor rate*) $\omega = 2\pi f(t)$



Instantaneous frequency



- Angular frequency ω (rate) is the derivative of the phase (the same way as the velocity *v(t)* is the derivative of distance *s(t)*)
- For continuously changing frequency instantaneous frequency is defined by differential changes:

$$\omega(t) = \frac{d\phi(t)}{dt} \quad \phi(t) = \int_{-\infty}^{t} \omega(\alpha)\, d\alpha$$

Compare to linear motion: $v(t) = \frac{ds(t)}{dt}\left(\approx \frac{s_2(t) - s_1(t)}{t_2 - t_1}\right)$

Frequency modulation (FM)

- In frequency modulation carrier instantaneous frequency is linearly proportional to modulation frequency:

$$\omega = 2\pi f(t) = d\theta_C(t)/dt$$
$$= 2\pi[f_C + f_\Delta x(t)]$$

- Hence the FM waveform can be written as

$$x_C(t) = A_c \cos(\underbrace{\omega_c t + 2\pi f_\Delta \int_{t_0}^{t} x(\lambda)d\lambda}_{\theta_C(t)}), t \geq t_0$$

$$\phi(t) = \int_{-\infty}^{t} \omega(\alpha)d\alpha$$

integrate

- Note that for FM

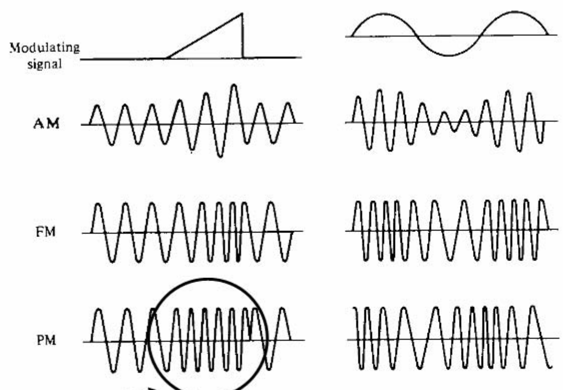
$$f(t) = f_C + f_\Delta x(t)$$

and for PM

$$\phi(t) = \phi_\Delta x(t)$$

	Instantaneous phase $\phi(t)$	Instantaneous frequency $f(t)$
PM	$\phi_\Delta x(t)$	$f_c + \frac{1}{2\pi}\phi_\Delta \dot{x}(t)$
FM	$2\pi f_\Delta \int^t x(\lambda)\,d\lambda$	$f_c + f_\Delta x(t)$

AM, FM and PM waveforms



Constant frequency: follows the derivative of the modulation waveform

$$x_{PM}(t) = A_c \cos(\omega_c t + \phi_\Delta x(t))$$
$$x_{FM}(t) = A_c \cos(\omega_c t + 2\pi f_\Delta \int^t x(\lambda)d\lambda)$$

	Instantaneous phase $\phi(t)$	Instantaneous frequency $f(t)$
PM	$\phi_\Delta x(t)$	$f_c + \frac{1}{2\pi}\phi_\Delta \dot{x}(t)$
FM	$2\pi f_\Delta \int^t x(\lambda)\,d\lambda$	$f_c + f_\Delta x(t)$

Narrowband FM and PM (small modulation index, arbitrary modulation waveform)

- The CW presentation: $x_c(t) = A_c \cos[\omega_c t + \phi(t)]$
- The quadrature CW presentation:

$$x_c(t) = x_{ci}(t)\cos(\omega_c t) - x_{cq}(t)\sin(\omega_c t)$$

$$x_{ci}(t) = A_c \cos\phi(t) = A_c[1 - (1/2!)\phi^2(t) + \ldots]$$

$$x_{cq}(t) = A_c \sin\phi(t) = A_c[\phi(t) - (1/3!)\phi^3(t) + \ldots]$$

- The narrow band condition: $|\phi(t)| << 1$rad

$$x_{ci}(t) \approx A_c \quad x_{cq}(t) \approx A_c\phi(t)$$

- Hence the Fourier transform of $X_C(t)$ is in this case

$$\mathbb{F}[x_C(t)] \approx \mathbb{F}[A_C\cos(\omega_C t) - A_C\phi(t)\sin(\omega_C t)]$$

$$X_C(f) \approx \frac{1}{2}A_C\delta(f - f_C) + \frac{j}{2}A_C\Phi(f - f_C), f > 0$$

$$\mathbb{F}[\cos(2\pi f_0 t)] = \frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$$

$$\mathbb{F}[\cos(2\pi f_0 t + \theta)x(t)] = \frac{1}{2}[X(f - f_0)\exp(j\theta) + jX(f + f_0)\exp(-j\theta)]$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

Narrow band FM and PM spectra

- Remember the instantaneous phase in CW presentation:

$$x_c(t) = A_c \cos[\omega_c t + \phi(t)]$$

$$\phi_{PM}(t) = \phi_\Delta x(t)$$

$$\phi_{FM}(t) = 2\pi f_\Delta \int_{t_0}^{t} x(\lambda)d\lambda, t \geq t_0$$

- The small angle assumption produces compact spectral presentation for both FM and AM:

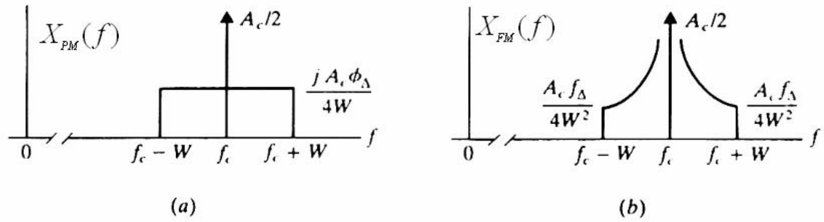
$$X_C(f) \approx \frac{1}{2}A_C\delta(f - f_C) + \frac{j}{2}A_C\Phi(f - f_C), f > 0$$

$$\Phi(f) = \mathbb{F}[\phi(t)] = \begin{cases} \phi_\Delta X(f), \text{PM} \\ -jf_\Delta X(f)/f, \text{FM} \end{cases}$$

What does it mean to set this component to zero?

$$\int_{t_0}^{t} g(\tau)d\tau \Leftrightarrow \frac{G(\omega)}{j\omega} + \pi G(0)\delta(\omega)$$

Example



- Assume: $x(t)=\mathrm{sinc}\,2Wt \Rightarrow X(f)=\frac{1}{2W}\Pi\left(\frac{f}{2W}\right)$

$$X_C(f)\approx\frac{1}{2}A_C\delta(f-f_C)+\frac{j}{2}A_C\Phi(f-f_C),\ f>0$$

$$\Phi_{PM}(f)=F[\phi_{PM}(t)]=\phi_\Delta X(f) \qquad \Phi_{FM}(f)=F[\phi_{FM}(t)]=-jf_\Delta X(f)/f$$

$$X_{PM}(f)\approx\frac{1}{2}A_C\delta(f-f_C)+\frac{j}{4W}A_C\phi_\Delta\Pi\left(\frac{f-f_C}{2W}\right),\ f>0$$

$$X_{FM}(f)\approx\frac{1}{2}A_C\delta(f-f_C)+\frac{f_\Delta}{4|f-f_C|W}A_C\Pi\left(\frac{f-f_C}{2W}\right),\ f>0$$

Tone modulation with PM and FM: modulation index β

- Remember the FM and PM waveforms:

$$x_{PM}(t)=A_c\cos[\omega_c t+\underbrace{\phi_\Delta x(t)}_{\phi(t)}]$$

$$x_{FM}(t)=A_c\cos[\omega_c t+\underbrace{2\pi f_\Delta\int_t x(\lambda)d\lambda}_{\phi(t)}]$$

- Assume tone modulation

$$x(t)=\begin{cases}A_m\sin(\omega_m t),\mathrm{PM}\\A_m\cos(\omega_m t),\mathrm{FM}\end{cases}$$

- Then

$$\phi(t)=\begin{cases}\phi_\Delta x(t)=\underbrace{\phi_\Delta A_m}_{\beta}\sin(\omega_m t),\mathrm{PM}\\2\pi f_\Delta\int_t x(\lambda)d\lambda=\underbrace{(A_m f_\Delta/f_m)}_{\beta}\sin(\omega_m t),\mathrm{FM}\end{cases}$$

Tone modulation in frequency domain: Phasors and spectra for narrowband case

- Remember the quadrature presentation:

$$x_c(t) = A_c \cos[\omega_c t + \phi(t)]$$
$$x_c(t) = x_{ci}(t)\cos(\omega_c t) - x_{cq}(t)\sin(\omega_c t)$$
$$x_{ci}(t) = A_c \cos\phi(t) = A_c[1-(1/2!)\phi^2(t)+\dots]$$
$$x_{cq}(t) = A_c \sin\phi(t) = A_c[\phi(t)-(1/3!)\phi^3(t)+\dots]$$

- For narrowband assume $\beta << 1, \phi(t) = \beta\sin(\omega_m t), \mathrm{FM}, \mathrm{PM}$

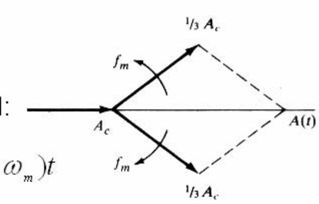
$$x_c(t) = A_c\cos(\omega_c t) - A_c\beta\sin(\omega_m t)\sin(\omega_c t)$$
$$= A_c\cos(\omega_c t) - \frac{A_c\beta}{2}\cos(\omega_c - \omega_m)t + \frac{A_c\beta}{2}\cos(\omega_c + \omega_m)t$$

$$\beta_{PM} = \phi_\Delta A_m$$
$$\beta_{FM} = A_m f_\Delta / f_m$$

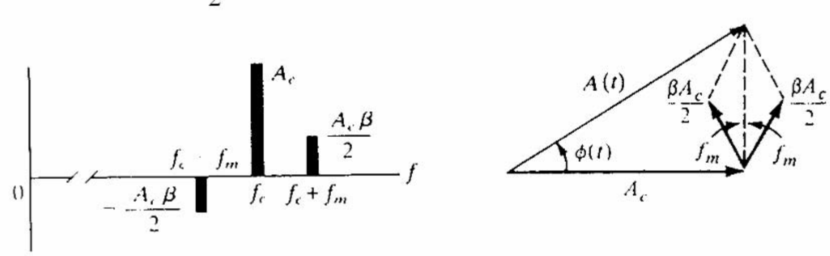
$$\sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

Narrow band tone modulation: spectra and phasors

- Phasors and spectra resemble AM:



$$x_c(t) = A_c\cos(\omega_c t) - \frac{A_c\beta}{2}\cos(\omega_c - \omega_m)t + \frac{A_c\beta}{2}\cos(\omega_c + \omega_m)t$$



(a) (b)

FM and PM with tone modulation and arbitrary modulation index

- Time domain expression for FM and PM:

$$x_c(t) = A_c \cos[\omega_c t + \beta \sin(\omega_m t)]$$

$$\beta_{PM} = \phi_\Delta A_m$$
$$\beta_{FM} = A_m f_\Delta / f_m$$

- Remember: $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- Therefore:

$$x_c(t) = A_c \cos(\beta \sin(\omega_m t))\cos(\omega_c t) - A_c \sin(\beta \sin(\omega_m t))\sin(\omega_c t)$$

$$\cos(\beta \sin(\omega_m t)) = J_o(\beta) + \sum_{n\,\mathrm{even}}^{\infty} 2J_n(\beta)\cos(n\omega_m t)$$
$$\sin(\beta \sin(\omega_m t)) = \sum_{n\,\mathrm{odd}}^{\infty} 2J_n(\beta)\sin(n\omega_m t)$$

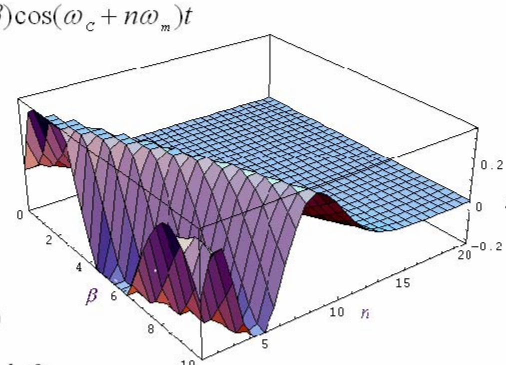
J_n is the first kind, order n Bessel function

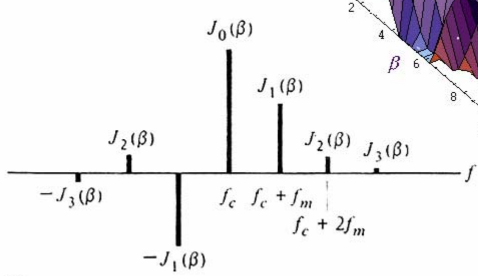
Wideband FM and PM spectra

- After simplifications we can write:

$$x_c(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\cos(\omega_c + n\omega_m)t$$

$$\beta = \begin{cases} \phi_\Delta A_m, \mathrm{PM} \\ A_m f_\Delta / f_m, \mathrm{FM} \end{cases}$$

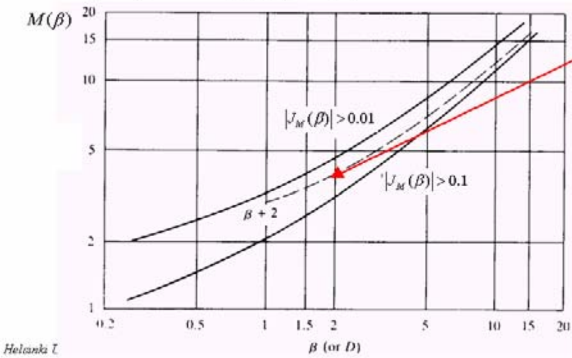




note: $J_{-n}(\beta) = (-1)^n J_n(\beta)$

Determination of transmission bandwidth

- The goal is to determine the number of significant sidebands
- Thus consider again how Bessel functions behave as the function of β, e.g. we consider $A_m \leq 1, f_m \leq W$
- Significant sidebands: $|J_n(\beta)| > \varepsilon$
- Minimum bandwidth includes 2 sidebands (why?): $B_{T\,min} = 2f_m$
- Generally: $B_T = 2M(\beta)f_m, M(\beta) \geq 1$



$M(\beta) \approx \beta + 2$

$$\beta = \begin{cases} \phi_\Delta A_m, \text{PM} \\ A_m f_\Delta / f_m, \text{FM} \end{cases}$$

Transmission bandwidth and deviation D

- Tone modulation is extrapolated into arbitrary modulating signal by defining deviation by

$$\beta = A_m f_\Delta / f_m \big|_{A_m=1, f_m=W} = f_\Delta / W \equiv D$$

- Therefore transmission BW is also a function of deviation

$$B_T = 2M(D)W$$

- For very large D and small D with $M(D) \approx D + 2$

$$B_T \approx 2(D+2)f_m \big|_{D>>1, f_m=W} \approx 2(D+2)W$$

$$\approx 2DW, D >> 1$$

$$B_T = 2M(D)W$$

$$\approx 2W, D << 1 \text{ (a single pair of sidebands)}$$

- that can be combined into

$$B_T = 2|D-1|W, D >> 1, \text{and } D << 1$$

$$\beta = \begin{cases} \phi_\Delta A_m, \text{PM} \\ A_m f_\Delta / f_m, \text{FM} \end{cases}$$

Example: Commercial FM bandwidth

- Following commercial FM specifications

$$f_\Delta = 75\,\text{kHz}, W \approx 15\,\text{kHz}$$
$$\Rightarrow D = f_\Delta / W = 5$$
$$B_T = 2(D+2)W \approx 210\,\text{kHz}, (D > 2)$$

- High-quality FM radios RF bandwidth is about

$$B_T \geq 200\,\text{kHz}$$

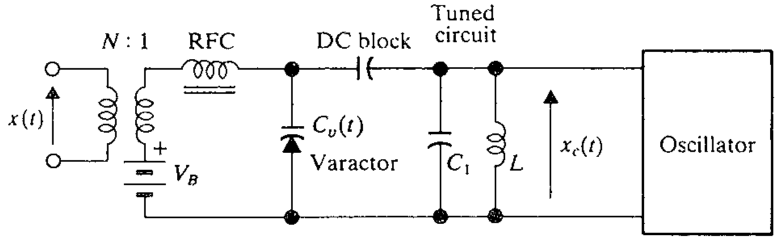
- Note that

$$B_T = 2|D-1|W \approx 180\,\text{kHz},\; D >> 1$$

under estimates the bandwidth slightly

A practical FM modulator circuit realization

- A tuned circuit oscillator
  - biased varactor diode capacitance directly proportional to *x(t)*
  - other parts:
    - input transformer
    - RF-choke
    - DC-block



Generating FM/PM

De-tuned tank circuit

- Capacitance of a resonant circuit can be made to be a function of modulation voltage.

$f_{\infty} = 1/(2\pi\sqrt{LC})$ Resonance frequency

$f_{\infty}[x(t)] = 1/\{2\pi\sqrt{LC[x(t)]}\}$ De-tuned resonance frequency

$C[x(t)] = C_0 + Cx(t)$ Capacitance diode

$f_{\infty}[x(t)] = f_c(1 - Cx(t)/C_0)^{-1/2}, f_c = 1/(2\pi\sqrt{LC_0})$

- That can be simplified by the series expansion

$$(1 - kx)^{-1/2} = 1 + \frac{kx}{2} + \frac{3k^2x^2}{8} \ldots |kx| << 1$$

$$f_{\infty}[x(t)] = f_c(1 - Cx(t)/C_0)^{-1/2}$$

$$\approx f_c\left[1 + \frac{1}{2}\frac{Cx(t)}{C_0}\right] = \frac{1}{2\pi}\frac{d\phi(t)}{dt}$$

$$\phi(t) = 2\pi f_c t + 2\pi \underbrace{\frac{C}{2C_0} f_c}_{f_\Delta} \int_t x(\lambda)d\lambda$$

Note that this applies for a relatively small modulation index

Remember that the instantaneous frequency is the derivative of the phase

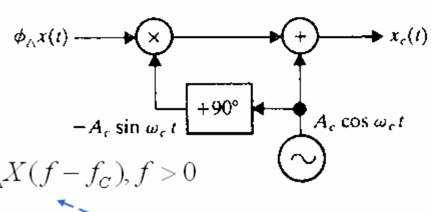
Phase modulators: narrow band mixer modulator

- Integrating the input signal to a phase modulator produces frequency modulation! Thus PM modulator applicable for FM

$$x_{PM}(t) = A_c \cos(\omega_c t + \phi_\Delta x(t))$$

$$x_{FM}(t) = A_c \cos(\omega_c t + 2\pi f_\Delta \int_t x(\lambda)d\lambda)$$

- Also, PM can be produced by an FM modulator by differentiating its input
- Narrow band mixer modulator:



$$X_C(f) \approx \frac{1}{2}A_C\delta(f - f_C) + \frac{j}{2}A_C\phi_\Delta X(f - f_C), f > 0$$

$$\begin{cases} \phi(t) = \phi_\Delta x(t) \text{ PM} \\ d\phi(t)/dt = 2\pi f(t) = 2\pi[f_C + f_\Delta x(t)] \text{ FM} \end{cases}$$

Narrow band FM/PM spectra

Indirect FM transmitter

- FM/PM modulator with high linearity and modulation index difficult to realize
- One can first generate a small modulation index signal that is then applied into a nonlinear circuit

Mathematica®-expressions

In[5]:= TrigReduce[Cos[ω_C t + β Sin[ω_m t]]^2]

Out[5]= $\frac{1}{2}(\cos(2\beta\sin(t\omega_m)+2t\omega_C)+1)$

In[6]:= TrigReduce[Cos[ω_C t + β Sin[ω_m t]]^3]

Out[6]= $\frac{1}{4}(3\cos(\beta\sin(t\omega_m)+t\omega_C)+\cos(3\beta\sin(t\omega_m)+3t\omega_C))$

In[7]:= TrigReduce[Cos[ω_C t + β Sin[ω_m t]]^4]

Out[7]= $\frac{1}{8}(4\cos(2\beta\sin(t\omega_m)+2t\omega_C)+\cos(4\beta\sin(t\omega_m)+4t\omega_C)+3)$

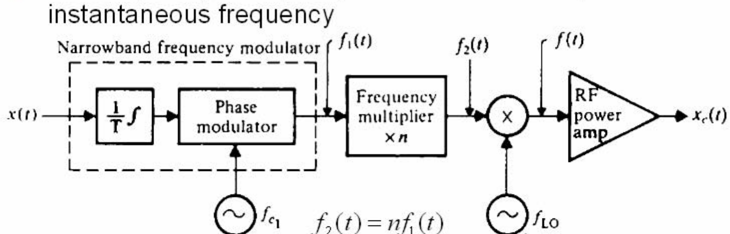
In[8]:= TrigReduce[Cos[ω_C t + β Sin[ω_m t]]^5]

Out[8]= $\frac{1}{16}(\underbrace{10\cos(\beta\sin(t\omega_m)+t\omega_C)+5\cos(3\beta\sin(t\omega_m)+3t\omega_C)}_{\text{should be filtered away, how?}}+\cos(5\beta\sin(t\omega_m)+5t\omega_C))$

- Therefore applying FM/PM wave into non-linearity increases modulation index

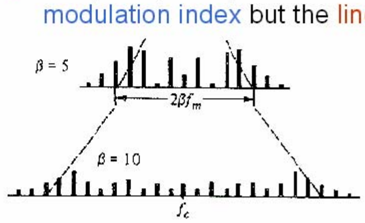
Indirect FM transmitter:circuit realization

- The frequency multiplier produces *n*-fold multiplication of instantaneous frequency



$$f_2(t) = nf_1(t) = nf_{c1} + n\phi_\Delta \quad \left(\text{FM: } n\phi_\Delta = 2\pi f_\Delta \int_t x(\lambda)d\lambda\right)$$

- Frequency multiplication of tone modulation increases modulation index but the line spacing remains the same



In[14]:= TrigReduce[Cos[ω_0 t + A_m Sin[ω_m t]]^2]

Out[14]= $\frac{1}{2}$ (1 + Cos[2 Sin[t ω_m] A_m + 2 t ω_0])

Frequency detection

- Methods of frequency detection
  - FM-AM conversion followed by envelope detector
  - Phase-shift discriminator
  - Zero-crossing detection (tutorials)
  - PLL-detector (tutorials)
- **FM-AM conversion** is produced by a transfer function having magnitude distortion, as the time derivative *(other possibilities?):*

$$x_c(t) = A_c \cos(\omega_0 t + \phi(t))$$

$$\frac{dx_c(t)}{dt} = -A_c \sin[\omega_c t + \phi(t)](\omega_c + d\phi(t)/dt)$$

$$d\phi(t)/dt = 2\pi f(t) = 2\pi[f_c + f_\Delta x(t)] \text{ FM}$$

- As for example

```
In[14]:= D[Cos[ω_c t + A_m Integrate[f[t], t]], t] // Expand

Out[14]= -f[t] Sin[(∫f[t] dt) A_m + t ω_c] A_m -
          Sin[(∫f[t] dt) A_m + t ω_c] ω_c
```