

S-72.1140 Transmission Methods in Telecommunication Systems (5 cr)

*Noise in analog carrier wave (CW) modulation
systems*

Noise in analog CW modulation systems

- Understanding noise
 - Lowpass presentation of bandpass noise and its conversion to baseband noise
 - Noise statistics of quadrature presentation in rectangular and polar coordinates
- Modeling detectors for linear and exponential modulation
- Analysis of post-detection SNR
 - Synchronous detector
 - PM-detector
 - FM-detector

Noise in carrier wave modulation systems: basic definitions

- Objectives: Define bandpass noise and use it to analyze post detection SNR of analog CW systems
- Assume signal is ergodic, e.g., all ensemble averages $E[\cdot]$ equal the corresponding time averages $\langle \cdot \rangle$. Then, for instance

$$\langle v(t) \rangle = E[v(t)] \quad \text{average value}$$

$$\langle v^2(t) \rangle = E[v^2(t)] \quad \text{average power}$$

$$\langle v(t)v(t - \tau) \rangle = E[v(t)v(t - \tau)] \quad \text{autocorrelation}$$

where the time average is defined by

$$\langle v_i(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} v_i(t) dt$$

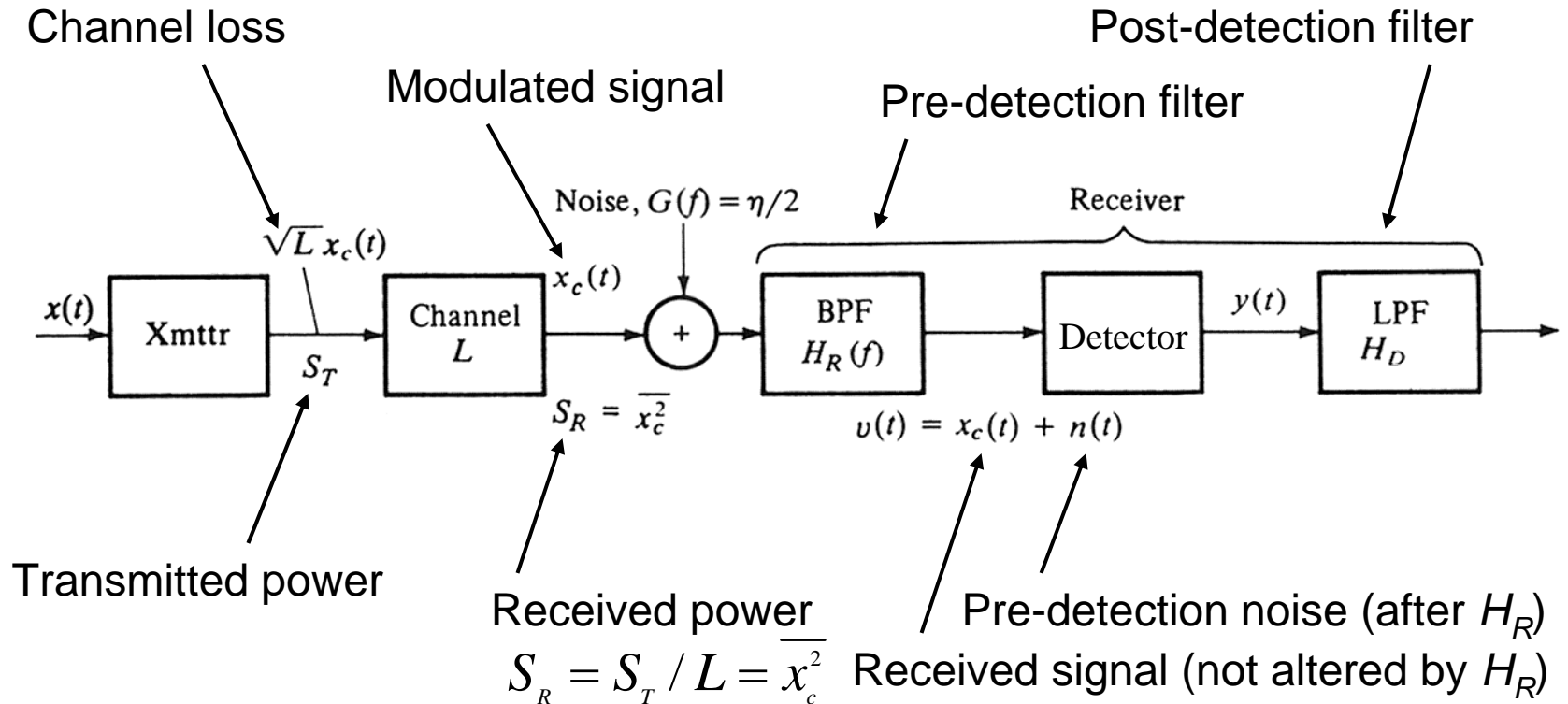
or

$$\langle v_i(t) \rangle = \frac{1}{T} \int_{-T/2}^{T/2} v_i(t) dt \quad (\text{for a known period})$$

The system model

- We consider normalized ergodic analog message whose amplitude and power are normalized

$$|x(t)| \leq 1, \quad S_x = \overline{x^2} = \langle x^2(t) \rangle, S_x \leq 1$$



Detection models

- Pre-detection signal $v(t)$ is presented in quadrature-carrier form:

$$\begin{aligned}
 v(t) &= A_v(t) \cos[\omega_c t + \phi_v(t)] & \begin{cases} v_i(t) = A_v(t) \cos \phi(t) \\ v_q(t) = A_v(t) \sin \phi(t) \end{cases} \\
 &= v_i(t) \cos(\omega_c t) - v_q(t) \sin(\omega_c t)
 \end{aligned}$$

- Detection models:

$$y(t) = \begin{cases} v_i(t) & \text{Synchronous detector} \\ A_v(t) - \overline{A_v} & \text{Envelope detector} \\ \phi_v(t) & \text{Phase detector} \\ d\phi_v(t)/dt & \text{Frequency detector} \end{cases}$$

(Remember that FM was defined by:

$$x_c(t) = A_c \cos[\omega_c t + 2\pi f_\Delta \int_{t_0}^t x(\lambda) d\lambda], t \geq t_0)$$

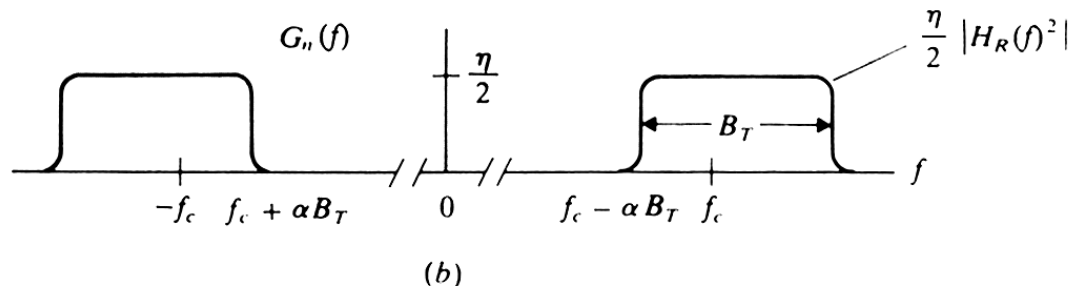
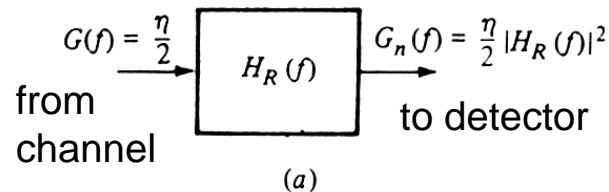
Pre-detection noise in bandpass channel

- Signal and noise are statistically independent and therefore their power can be added to form the total pre-detection power:

$$\overline{v^2} = \overline{x_c^2} + \overline{n^2} = S_R + N_R$$

- The pre-detection (bandpass) noise power is filtered from the channel noise:

$$N_R = \int_{-\infty}^{\infty} (\eta/2) |H_R(f)|^2 df = 2 \int_0^{B_T} (\eta/2) df = \eta B_T$$



Pre-detection SNR

- Pre-detection signal-to-noise ratio for **bandpass** channels is defined by

$$S_R / N_R = S_R / (\eta B_T)$$

- Note that above B_T is the transmission bandwidth passing channel noise power to the detector

$$N_R = \eta B_T = (\eta / 2) \int_{-\infty}^{\infty} |H_R(f)|^2 df$$

- For comparison, we can write the received signal-to-noise in terms of **baseband system** (BW = W) SNR defined by

$$\gamma = S_R / (\eta W)$$

and therefore also

$$S_R / N_R = S_R / (\eta B_T) = (S_R / \eta W)(W / B_T) = \gamma W / B_T$$

- Note that always (limiting case is the SSB with $B_T = W$)

$$B_T \geq W \Rightarrow S_R / N_R \leq \gamma$$

(We will see, however, that post detection SNR can be much larger than γ !)

Bandpass noise: $n(t) = A_n(t) \cos[\omega_c t + \phi_n(t)]$

- We assume stationary, zero mean Gaussian noise process for which

$$\overline{n} = 0, \overline{n^2} = \sigma_n^2 = N_R$$

- Bandpass noise in terms of lowpass equivalent signals

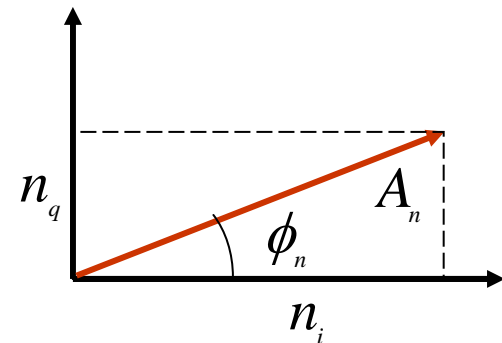
$$n(t) = n_i(t) \cos(\omega_c t) - n_q(t) \sin(\omega_c t)$$

- The in-phase and quadrature components are independent and hence

$$\overline{n_i(t)n_q(t)} = 0$$

- Their average is zero $\overline{n_i} = \overline{n_q} = 0$ and their average power is the same:

$$\overline{n_i^2} = \overline{n_q^2} = \overline{n^2} = N_R$$



Bandpass noise has Rayleigh distributed envelope and evenly distributed phase

- I-Q components of the bandpass noise can be presented in envelope - phase format:

$$\overline{A_n^2} = \overline{n_i^2} + \overline{n_q^2}, \phi_n = \arctan \frac{n_q}{n_i}$$

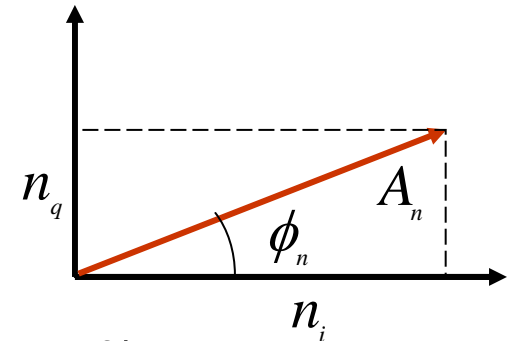
Two independent r.v.:s - sum of their variances equals variance of the envelope

- The PDF of envelope is Rayleigh distributed defined by

$$p_{A_n}(A_n) = \frac{A_n^2}{N_R} \exp\left(-\frac{A_n^2}{2N_R}\right) u(A_n)$$

- Therefore mean and variance for the bandpass noise are (integrate from above, how?)

$$\begin{aligned} \overline{A_n} &= \sqrt{\pi N_R / 2}, \quad \overline{A_n^2} = 2N_R \\ \Rightarrow N_R &= \overline{A_n^2} / 2 = (\overline{n_i^2} + \overline{n_q^2}) / 2 \Big|_{\overline{n_i^2} = \overline{n_q^2} = \overline{n^2}} = \overline{n^2} \end{aligned}$$



Post detection noise in synchronous detection

- **Signal component** of synchronous detector:

$$v_{DSB}(t) = x(t) \exp(j\omega_c t)$$

received DSB signal

$$v_{DSB}(t) = A_c x(t) \cos(\omega_c t) + jA_c x(t) \sin(\omega_c t)$$

$$v_{DSB}(t) \cos(\omega_c t) = \frac{A_c x(t)}{2} + \frac{A_c x(t)}{2} \cos(2\omega_c t) + j \frac{A_c x(t)}{2} \sin(2\omega_c t)$$

← detected message

- **Noise component** of synchronous detector:

$$n(t) \cos(\omega_c t) = \cos(\omega_c t) [n_i(t) \cos(\omega_c t) + n_q(t) \sin(\omega_c t)]$$

$$= n_i(t) [1 + \cos(2\omega_c t)] / 2 + n_q(t) \sin(2\omega_c t) / 2$$

- Detector extracts i -components and removes double frequency components

$$y_D(t) = A_c x(t) + n_i(t)$$

$$\begin{aligned} \cos(x) \cos(x) &= 1/2 + \cos(2x)/2 \\ \cos(x) \sin(x) &= \sin(2x)/2 \end{aligned}$$

Post-detection SNR for DSB

- Obtain signal and noise power **after detection** from:

$$y_D(t) = A_c x(t) + n_i(t)$$

where average noise and signal power are

$$N_D = \langle n_i^2(t) \rangle, S_D = \langle A_c^2 x^2(t) \rangle = A_c^2 S_x$$

Received average signal power is

$$S_R = \langle A_c^2 \underbrace{x^2(t)}_{S_x} \underbrace{\cos^2(\omega_0 t)}_{1/2} \rangle = A_c^2 S_x / 2 \Rightarrow S_x = 2S_R / A_c^2$$

and therefore SNR after DSB detector is

$$\begin{aligned} S_D / N_D &= A_c^2 S_x / N_D \\ &= \frac{\cancel{A_c^2} 2S_R}{\cancel{A_c^2} N_D} = \frac{2S_R}{N_D} = \frac{2S_R}{\eta B_T} \Big|_{B_T=2W \text{ (DSB)}} = \frac{S_R}{\eta W} = \gamma \text{ (DSB)} \end{aligned}$$

Comparing SNR for DSB and AM

- It can be show, that for AM the post detection SNR is

$$S_D / N_D |_{AM} = \gamma \frac{S_x}{1 + S_x}$$

- Comparison of this to the SNR of DSB can done by noting that in practice

$$S_x = 0.5 \text{ tone modulation}$$

$$S_x \approx 0.1 \text{ speech signal}$$

- Hence AM performs usually much worse than DSB
- It can be shown that for SSB performance is the same as for DSB, e.g.

$$S_D / N_D |_{USB,DSB} = \frac{S_R}{\eta W} = \gamma$$

Exponential modulation and channel noise

- Both PM and FM have constant envelopes so the received power is constant

$$x_c(t) = A_c \cos[\omega_c t + \phi(t)]$$

$$S_R = \overline{x_c^2} = A_c^2 / 2$$

- Received SNR** is $\frac{S_R}{N_R} = \frac{A_c^2}{2N_R} = \frac{A_c^2}{2\eta B_T}$ yielding for wideband FM

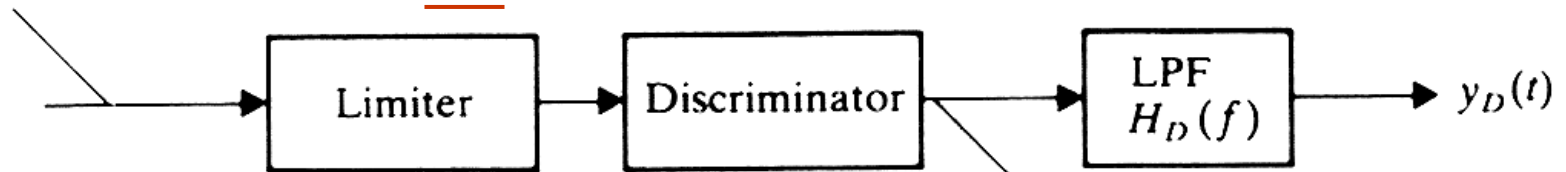
$$\frac{S_R}{N_R} = \frac{S_R}{\eta B_T} = \frac{S_R}{\eta 2DW} = \frac{\gamma}{2D}$$

where for wideband modulation

$$\begin{aligned} B_T &\approx 2(\beta + 2) f_m \Big|_{\beta \gg 1, f_m = W} \approx 2\beta W \\ &= 2DW \end{aligned}$$

Detection of exponential modulation assuming small noise power

$$v(t) = x_c(t) + n(t) = A_v(t) \cos [\omega_c t + \phi_v(t)]$$



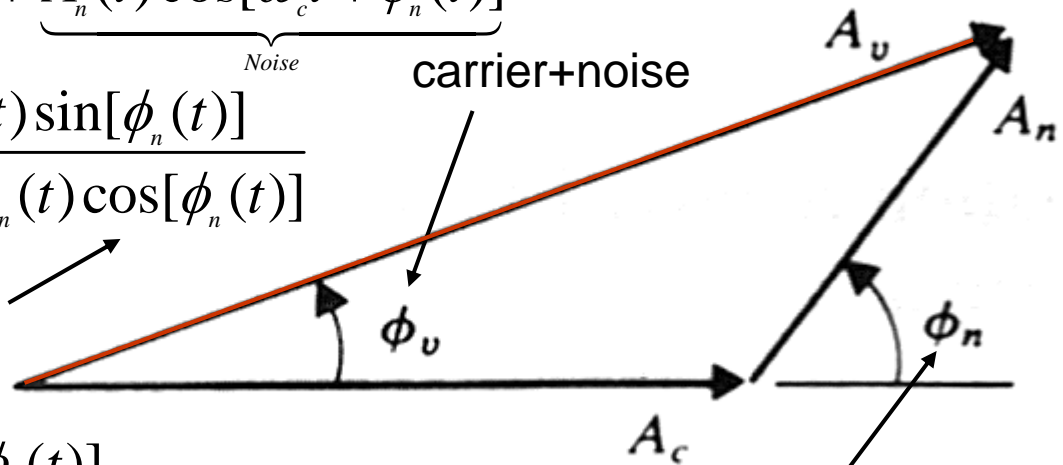
$$\left(\frac{S}{N}\right)_R = \frac{A_c^2/2}{\eta B_T}$$

$$y(t) = \begin{cases} \phi_v(t) & \text{PM} \\ \frac{1}{2\pi} \dot{\phi}_v(t) & \text{FM} \end{cases}$$

$$v(t) = \underbrace{A_c \cos[\omega_c t + \phi(t)]}_{\text{Signal}} + \underbrace{A_n(t) \cos[\omega_c t + \phi_n(t)]}_{\text{Noise}}$$

$$\phi_v(t) = \arctan \frac{A_n(t) \sin[\phi_n(t)]}{A_c + A_n(t) \cos[\phi_n(t)]}$$

small compared to A_c



$$\phi_v(t) \approx \frac{A_n(t) \sin[\phi_n(t)]}{A_c}$$

Detected noise component

Post-detection noise spectra for PM

$$\phi_v(t) \approx \frac{A_n(t) \sin \phi_n(t)}{A_c} \quad \begin{cases} S_R = A_c^2 / 2 \\ n_q = A_n \sin \phi_n \\ \overline{n_i^2} = \overline{n_q^2} = N_R = \eta B_{LP} \end{cases}$$

$$= \frac{n_q(t)}{\sqrt{2S_R}}$$

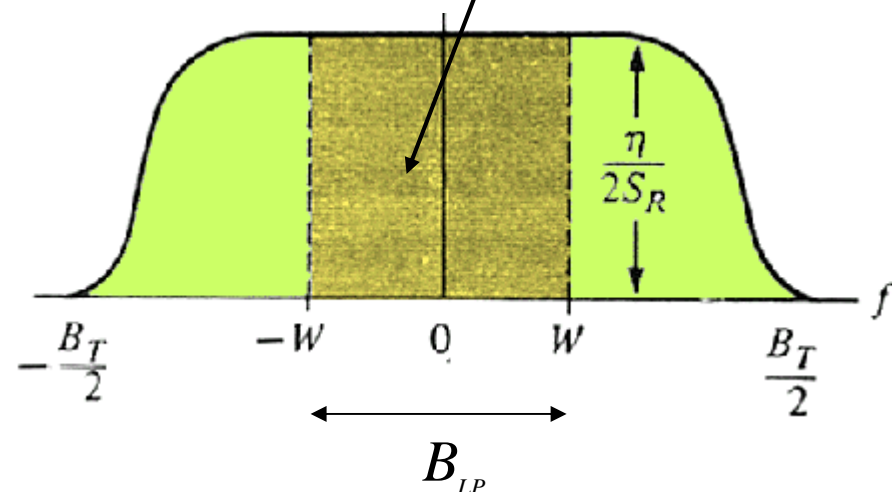
Note that after detection signal bandwidth is W and thus a post detection filter is required to remove out-of-band channel noise

- The channel noise is bandpass noise filtered at the transmission bandwidth and therefore the respective post-detection noise power spectral density $G_{PM}(f)$ and the total noise power N_D are

$$G_{PM}(f) = \left(\frac{n_q(t)}{\sqrt{2S_R}} \right)^2 \Pi \left(\frac{f}{B_T} \right)$$

$$= \frac{\eta}{2S_R} \Pi \left(\frac{f}{B_T} \right)$$

$$N_D = \int_{-W}^W G_{PM}(f) df = \eta W / S_R$$



Post-detection SNR for FM

- Recall the definition of FM-signal

$$x_c(t) = A_C \cos[\omega_C t + \phi_v(t)], \phi_v(t) = 2\pi f_\Delta \int_{t_0}^t x(\lambda) d\lambda$$

- Frequency discriminator (detector) differentiates the instantaneous phase to cancel out the inherent integration in phase. Now $\omega(t) = 2\pi x(t) = d\phi_v(t) / dt \Rightarrow$

$$x(t) = \frac{d\phi_v(t)}{2\pi dt} = \frac{1}{2\pi} \left\{ \underbrace{\frac{d\phi_s(t)}{dt}}_{\text{Signal}} + \underbrace{\frac{d\phi_N(t)}{dt}}_{\text{Noise}} \right\} = \frac{1}{2\pi} \left\{ \frac{d\phi_s(t)}{dt} + \frac{d[n_q(t)]}{\sqrt{2S_R} dt} \right\}$$

- Inspection in frequency domain (In order to find the respective PSDs) yields after detector

$$X(f) = j2\pi f \Phi_v(f)$$

and the signal PSD is

$$G_{FM}(f) = \left| \hat{X}(f) \right|^2 = (\cancel{2\pi} f)^2 |\Phi_v(f)|^2 / (\cancel{2\pi})^2 = f^2 |\Phi_v(f)|^2$$

$$\phi_v(t) \approx \frac{A_n(t) \sin \phi_n(t)}{A_c} = \frac{n_q(t)}{\sqrt{2S_R}} \qquad \frac{d^n x(t)}{dt^n} \leftrightarrow (j2\pi f)^n X(f)$$

Post-detection SNR in FM (cont.)

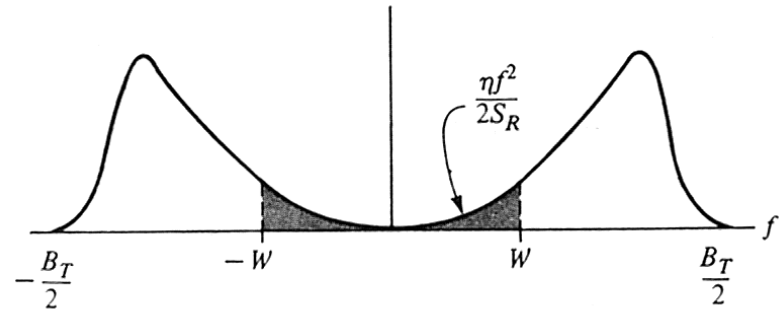
- Therefore, the post-detection noise PSD can be written as

$$G_{FM}(f) = f^2 |\Phi_v(f)|^2 \text{ with}$$

$$\Phi_v(f) = \frac{N_q(f)}{\sqrt{2S_R}}$$

and now the PSD for FM post detection noise is

$$G_{FM}(f) = \frac{\eta f^2}{2S_R} \Pi\left(\frac{f}{B_T}\right)$$



and the respective total noise power is

$$N_D = \int_{-W}^W G_{FM}(f) df = \eta W^3 / (3S_R)$$

Destination S/N for PM and FM

- For PM we have

$$S_D / N_D = \frac{\phi_\Delta^2 S_x}{\eta W / S_R} = \phi_\Delta^2 S_x \gamma, \text{ where } \phi_\Delta^2 S_x \leq \pi^2$$

- For FM we have

$$\begin{aligned} S_D / N_D &= \frac{f_\Delta^2 S_x}{\eta W^3 / (3S_R)} \\ &= 3 \underbrace{\left(\frac{f_\Delta}{W} \right)^2}_D S_x \underbrace{\frac{S_R}{\eta W}}_\gamma = \underline{\underline{3D^2 S_x \gamma}} \end{aligned}$$

- Under wideband condition $D \gg 1$ and

$$\begin{aligned} B_T &\approx 2(\beta + 2) f_m \Big|_{\beta \gg 1, f_m = W} \approx 2\beta W \\ &= 2DW \Rightarrow D = B_T / (2W) \end{aligned}$$

$$S_D / N_D = \underline{\underline{\frac{3}{4} \left(\frac{B_T}{W} \right)^2 S_x \gamma}}$$

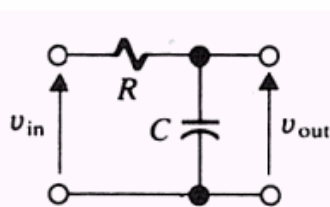
$$\begin{aligned} N_{D(PM)} &= \int_{-W}^W \frac{\eta}{2S_R} df = \frac{\eta W}{S_R} \\ N_{D(FM)} &= \int_{-W}^W \frac{\eta f^2}{2S_R} df = \frac{\eta W^3}{3S_R} \end{aligned}$$

Note that S_D/N_D can be increased just by increasing deviation!

FM preemphases and deemphases filters

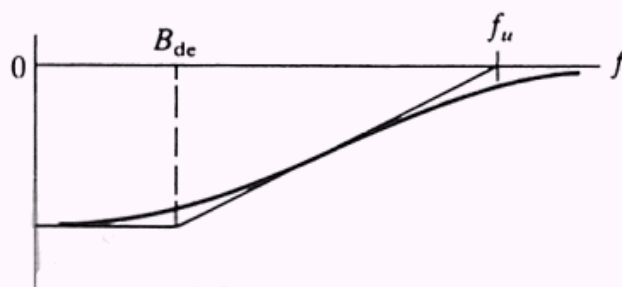
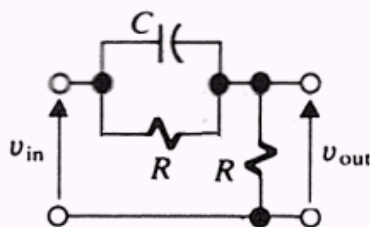
- FM related noise emphases can be suppressed by *pre-distortion* and post detection filters (preemphases and deemphases filters):

receiver filter



$$H_{de}(f)$$

transmitter filter



$$H_{pe}(f)$$

Q: What would happen if the filters would be reversed? (TX filter in receiver & vice versa)

$$H_{de}(f) = [1 + j(f / B_{de})]^{-1} \approx \begin{cases} 1, & |f| \ll B_{de} \\ B_{de} / (jf), & |f| \gg B_{de} \end{cases} \text{LPF}$$

$$H_{pe}(f) = [1 + j(f / B_{de})] \approx \begin{cases} 1, & |f| \ll B_{de} \\ j(f / B_{de}), & |f| \gg B_{de} \end{cases} \text{HPF}$$

FM post-detection S/N with deemphases

- Deemphases filter (that is a lowpass filter connected after detector) can suppress noise further. FM post-detection noise PSD and total noise power without deemphases:

$$G_{FM}(f) = \frac{\eta f^2}{2S_R} \Pi\left(\frac{f}{B_T}\right) \quad N_D = \int_{-W}^W G_{FM}(f) df = \frac{\eta W^3}{3S_R}$$

- With deemphases filter (for simplification assume $W/B_{de} \gg 1$):

$$N_D = \int_{-W}^W G_{FM}(f) |H_{de}(f)|^2 df = \frac{\eta B_{de}^3}{S_R} \left[\frac{W}{B_{de}} - \underbrace{\arctan \frac{W}{B_{de}}}_{\ll W/B_{de}} \right] \approx \eta B_{de}^2 W / S_R$$

where

$$|H_{de}(f)|^2 = \frac{1}{1 + (f / B_{de})^2}$$

$$S_D / N_D = \frac{f_\Delta^2 S_x}{\eta B_{de}^2 W / S_R} = \frac{S_x S_R}{\eta W} \left(\frac{f_\Delta}{B_{de}} \right)^2 = \underline{S_x \gamma \left(\frac{f_\Delta}{B_{de}} \right)^2}$$

Example

- FM radio

$$f_{\Delta} = 75 \text{ kHz}, W = 15 \text{ kHz}, D = 5, S_x = 1/2, B_{de} = 2.1 \text{ kHz}$$

- Without deemphases

$$\begin{aligned} S_D / N_D &= 3D^2 S_x \gamma \\ &= (3 \times 5^2 \times 1/2) \gamma = 38\gamma \end{aligned} \quad \gamma = \frac{S_R}{\eta W}$$

- With deemphases

$$S_D / N_D = (f_{\Delta} / B_{de})^2 S_x \gamma \approx 640\gamma$$

- Therefore if DSB or SSB system could be exchanged to FM system 640 fold transmission power savings could be achieved. Note, however that the required transmission bandwidth is now about $220 \text{ kHz} / 15 \text{ kHz} = 15$ times larger! Also, a problem is the FM threshold effect that we discuss next.

Comparison of carrier wave modulation systems

Type	$b = B_T/W$	$(S/N)_D/\gamma$	γ_{th}	DC	Complexity	Comments	Typical applications
Baseband	1	1	...	No†	Minor	No modulation	Short-haul links
AM	2	$\frac{\mu^2 S_x}{1 + \mu^2 S_x}$	20	No	Minor	Envelope detection $\mu \leq 1$	Broadcast radio
DSB	2	1	...	Yes	Major	Synchronous detection	Analog data, multiplexing
SSB	1	1	...	No	Moderate	Synchronous detection	Point-to-point voice, multiplexing
VSF	1+	1	...	Yes	Major	Synchronous detection	Digital data
VSF + C	1+	$\frac{\mu^2 S_x}{1 + \mu^2 S_x}$	20	Yes‡	Moderate	Envelope detection $\mu < 1$	Television video
PM§	$2M(\phi_\Delta)$	$\phi_\Delta^2 S_x$	$10b$	Yes	Moderate	Phase detection $\phi_\Delta \leq \pi$	Digital data
FM§¶	$2M(D)$	$3D^2 S_x$	$10b$	Yes	Moderate	Frequency detection	Broadcast radio, microwave relay, satellite systems

† Unless direct-coupled.

‡ With electronic DC restoration.

§ $b \geq 2$.

¶ Deemphasis not included.