Linear modulation methods

Modulation/demodulation (1)

Why modulate?

To match the signal to the available transmission channel

What is modulation/demodulation?

Linear/non-linear up-conversion/down-conversion of the information-bearing signal

What characterises a linear modulation?

The spectral shape of the modulating signal is preserved, it is only shifted to the carrier frequency

The bandwidth of the modulate signal is typically $B = 2W_x$

The modulated signal envelope varies as a function of time

$$s(t) = \mathrm{Re}\left\{z(t)\exp\left(j2\pi f_c t\right)\right\} = z_p(t)\cos\left(2\pi f_s t\right) - z_q(t)\sin\left(2\pi f_c t\right)$$

$z(t) = z_p(t) + jz_q(t)$ **is the complex modulating signal**

The later expression gives the quadrature representation

Modulation in radio communication systems

Important considerations

- **bandwidth efficiency, spectrum efficiency**
- **noise interference tolerance**
- **nonlinear amplification tolerance**
- **implementation factors**

Reference receiver error performance in AWGN-channel, assumptions:

- **additive white zero-mean Gaussian noise**
- **the receiver filter is matched to the noiseless *received* digital pulse shape or correlation is performed with the noiseless *received* digital pulse shape**
- **single symbol or ISI-free transmission**

Problems when applying reference receiver performance in radio systems

- **multipath propagation ⇒ ISI present**
- **dynamic channel ⇒ channel estimation needed for matched filtering**
- **AWGN-model not valid for radio interference**
- **carrier and symbol timing recovery methods impact performance**
- **received power variations ⇒ average BER, BER-statistics important**

Modulation/demodulation (2)

What characterises a non-linear modulation?

- **The spectral shape of the modulating signal is changed**
- **The bandwidth of the modulate signal is typically $B > 2W_x$**
- **The modulated signal envelope is mostly constant**

$$s(t) = \mathrm{Re}\left\{\sqrt{2P_{tx}}\exp\left(j2\pi f_c t\right)\exp\left(jf\left(x(t)\right)\right)\right\} = \sqrt{2P_{tx}}\cos\left(2\pi f_c t + f\left(x(t)\right)\right)$$

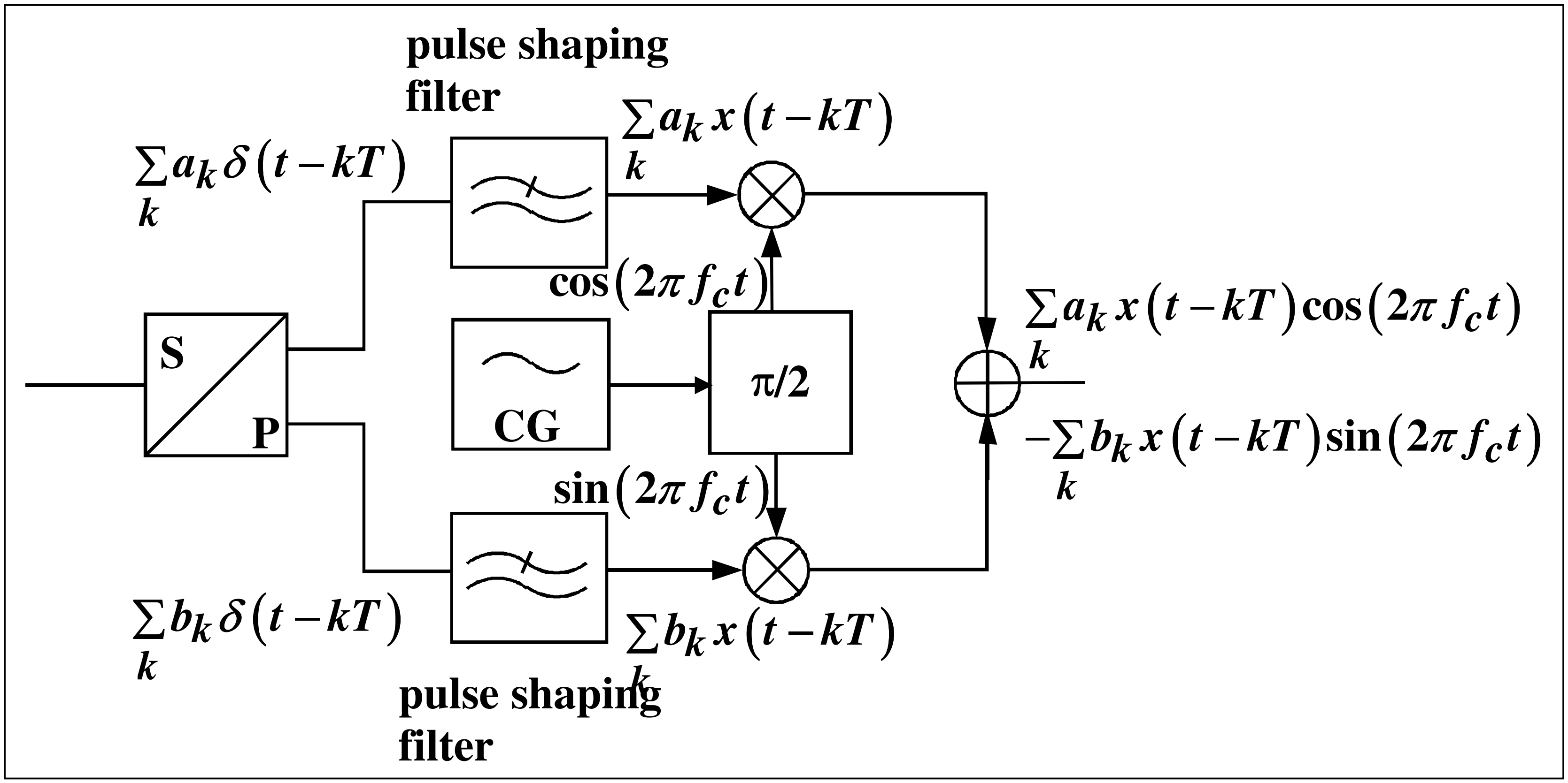
where $f(\;)$ mostly is a linear function (integration, convolution)

Generalized modulation $s(t) = \mathrm{Re}\left\{z(t)\exp\left(j2\pi f_c t\right)\exp\left(jf\left(x(t)\right)\right)\right\}$ contains both linear and non-linear modulation

Every modulation can be generated by a quadrature modulator, where the modulating signal is an unprocessed/linearly/non-linearly processed information signal

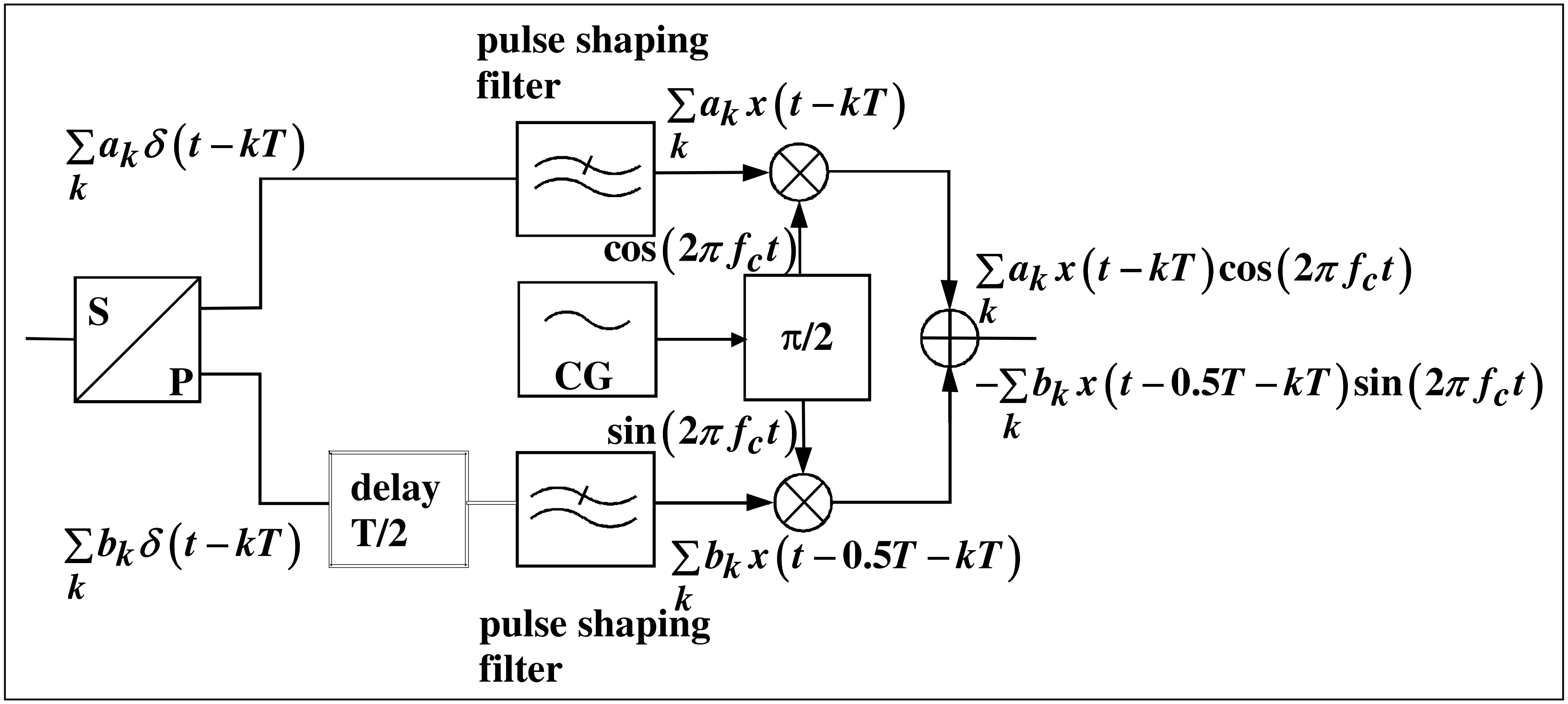
Modulation/demodulation (3)

The basic digital quadrature modulator



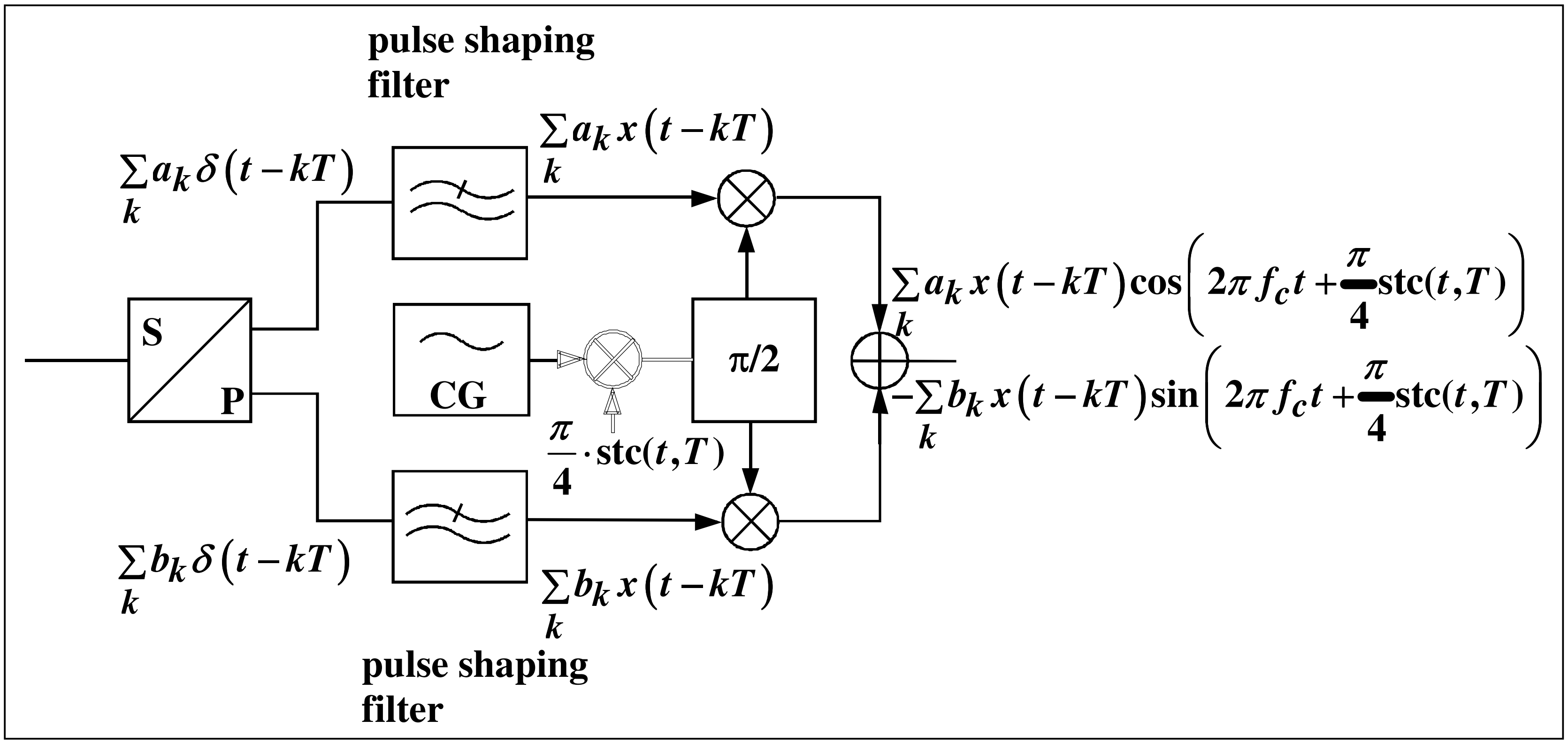
Modulation/demodulation (4)

The offset digital quadrature modulator



Modulation/demodulation (5)

The π/4-shifted digital quadrature modulator



The staircase function $\mathrm{stc}(t,T)=\sum_{k=k_0}^{k_1} \mathrm{u}(t-kT)$

- **QAM-methods (Quadrature Amplitude Modulation)**

QAM is a linear modulation where the expression for the modulated signal is:

$$s(t) = \mathrm{Re}\left\{ \sum_{k=-\infty}^{\infty} (a_k + jb_k) x(t-kT) e^{j2\pi f_c t} \right\}$$

$$= \sum_{k=-\infty}^{\infty} \left[a_k x(t-kT)\cos(2\pi f_c t) - b_k x(t-kT)\sin(2\pi f_c t) \right] \quad (2)$$

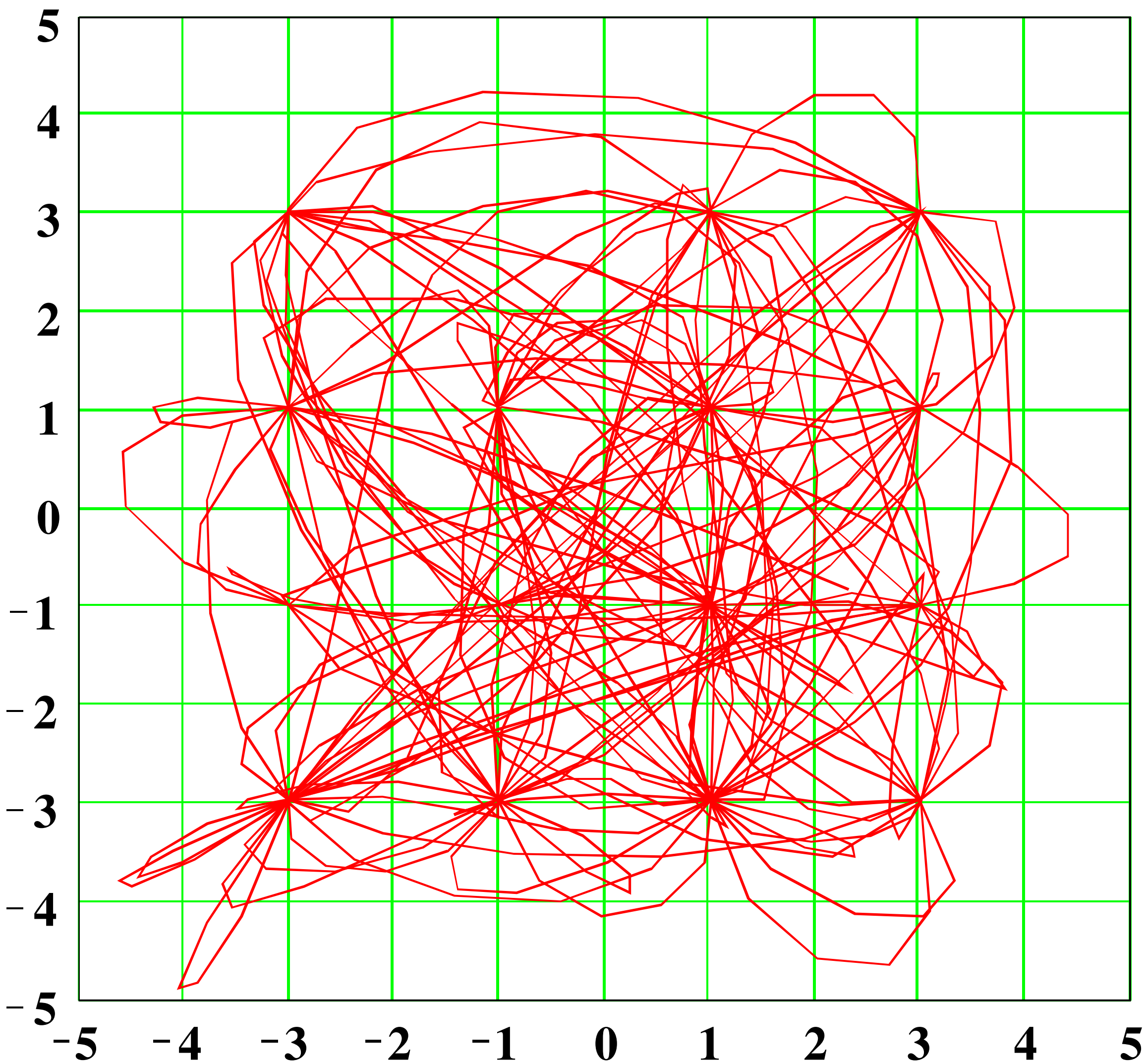
where

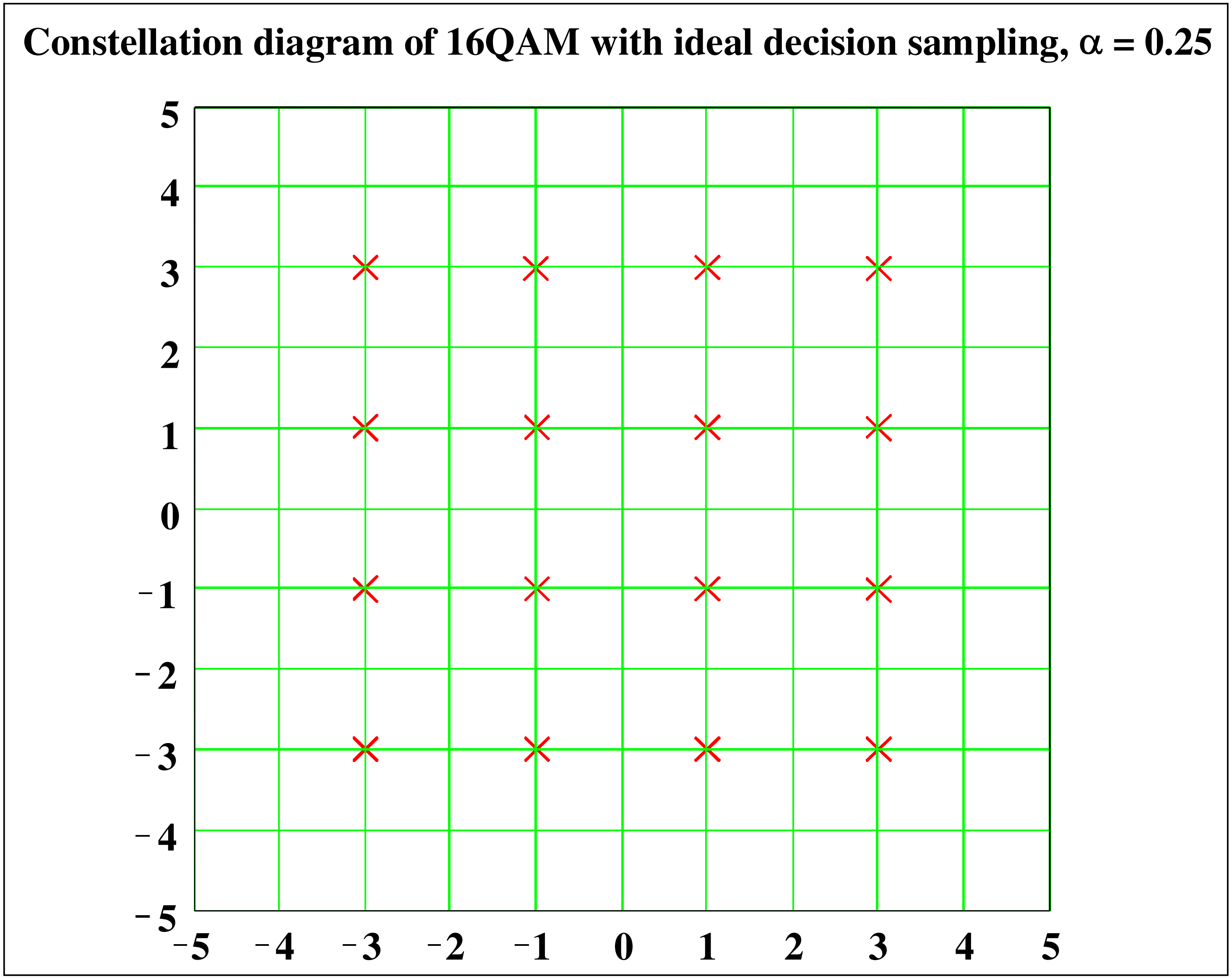
- $a_k,\ b_k = \pm 1, \pm 3, \ldots \pm (M-1)$, *where* $P(a_k) = P(b_k) = 1/M$

  $E\{a_k b_k\} = E\{a_k\} E\{b_k\} = 0, \quad E\{a_k a_l\} = E\{b_k b_l\} = \delta_{kl}$

- $x(t)$ **is the base-band pulse.**

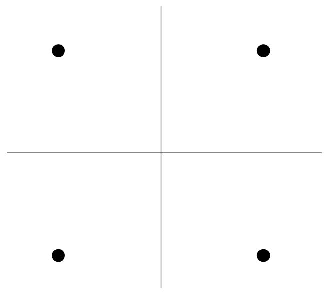
Phasor diagram of 16QAM with raised cosine filtering, α = 0.25



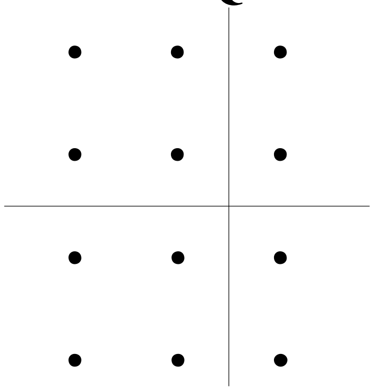

Constellation diagram of 16QAM with ideal decision sampling, α = 0.25
5
4
3
2
1
0
−1
−2
−3
−4
−5
−5 −4 −3 −2 −1 0 1 2 3 4 5

Basic QAM-constellations having about same average power

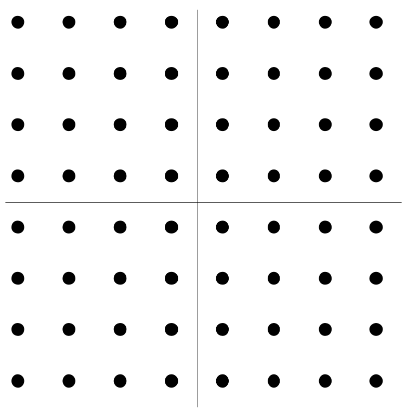
4QAM



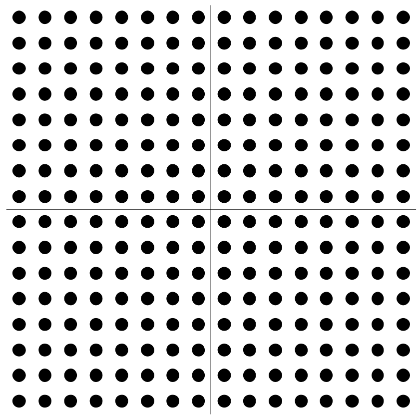
16QAM



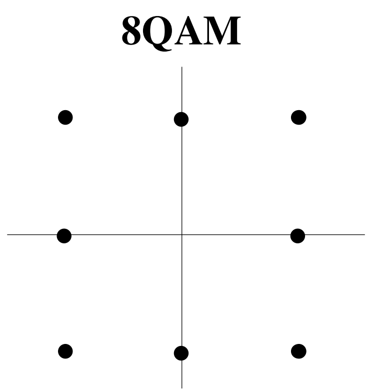
64QAM

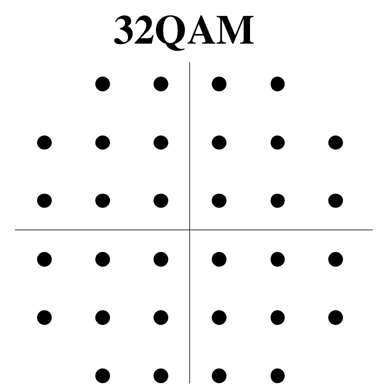


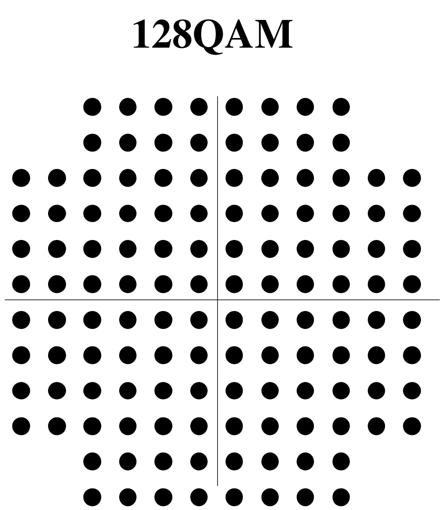
256QAM

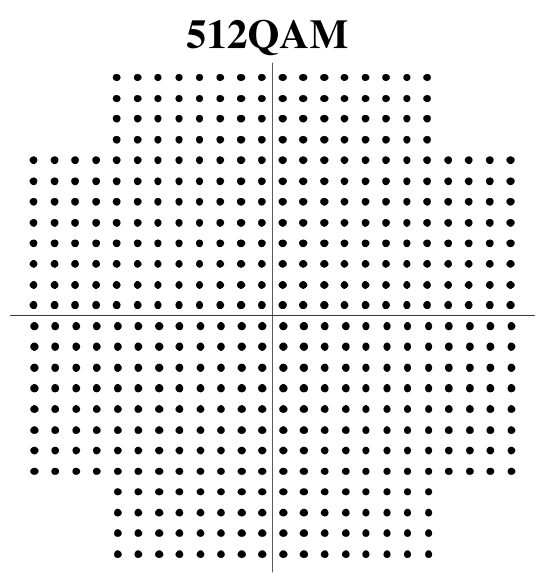


Examples of non-quadratic QAM-constellations



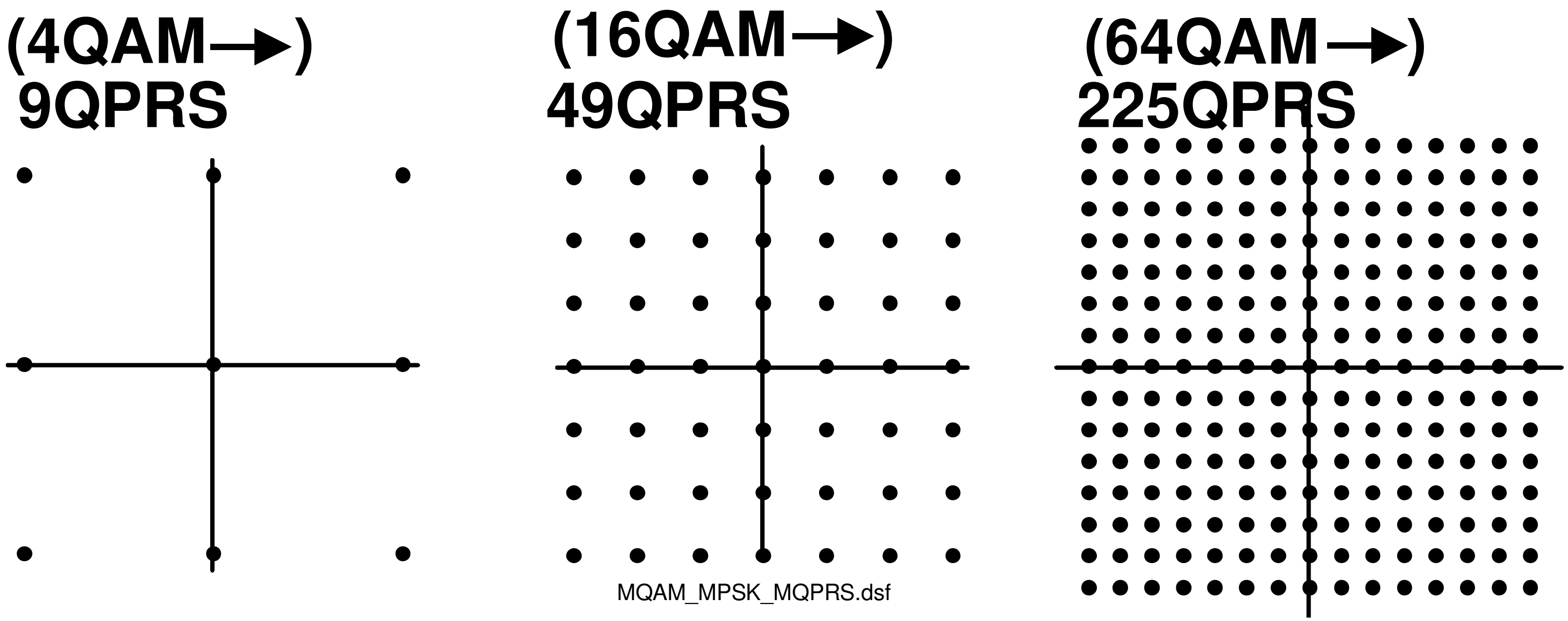






QPRS-constellations

(Quadrature Partial Response Signaling)



MPSK. (M-ary Phase Shift Keying)

Phase modulation representation:

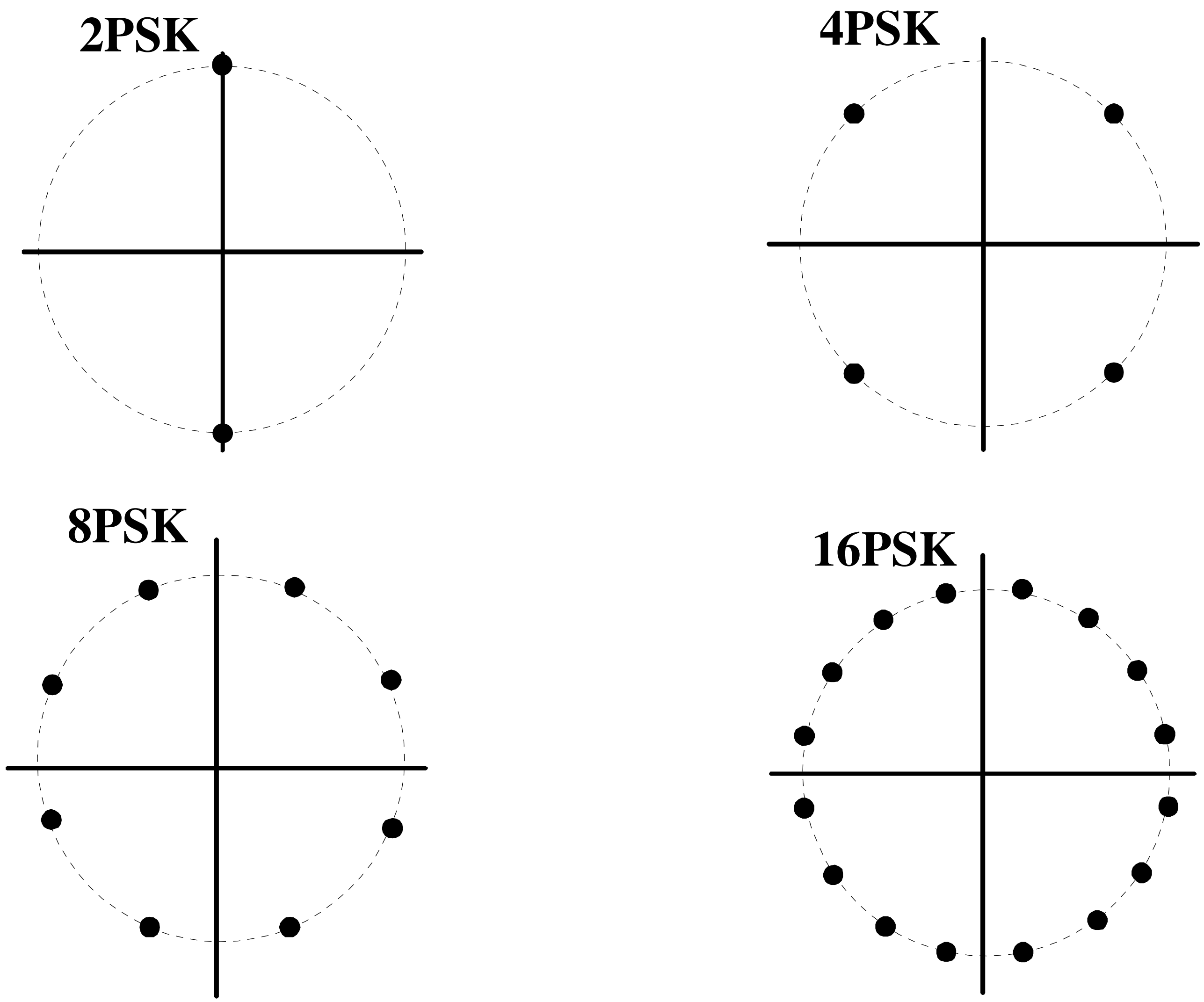
$$s(t) = \mathrm{Re}\left\{ \sqrt{2P_c} \exp\left(j(2\pi f_c t + \Delta\Phi \sum_{k=-\infty}^{\infty} a_k x(t - kT)) \right) \right\}$$

$$x(t) = rect\left(\frac{t - 0.5T}{T} \right), \quad \Delta\phi = \frac{\pi}{M}$$

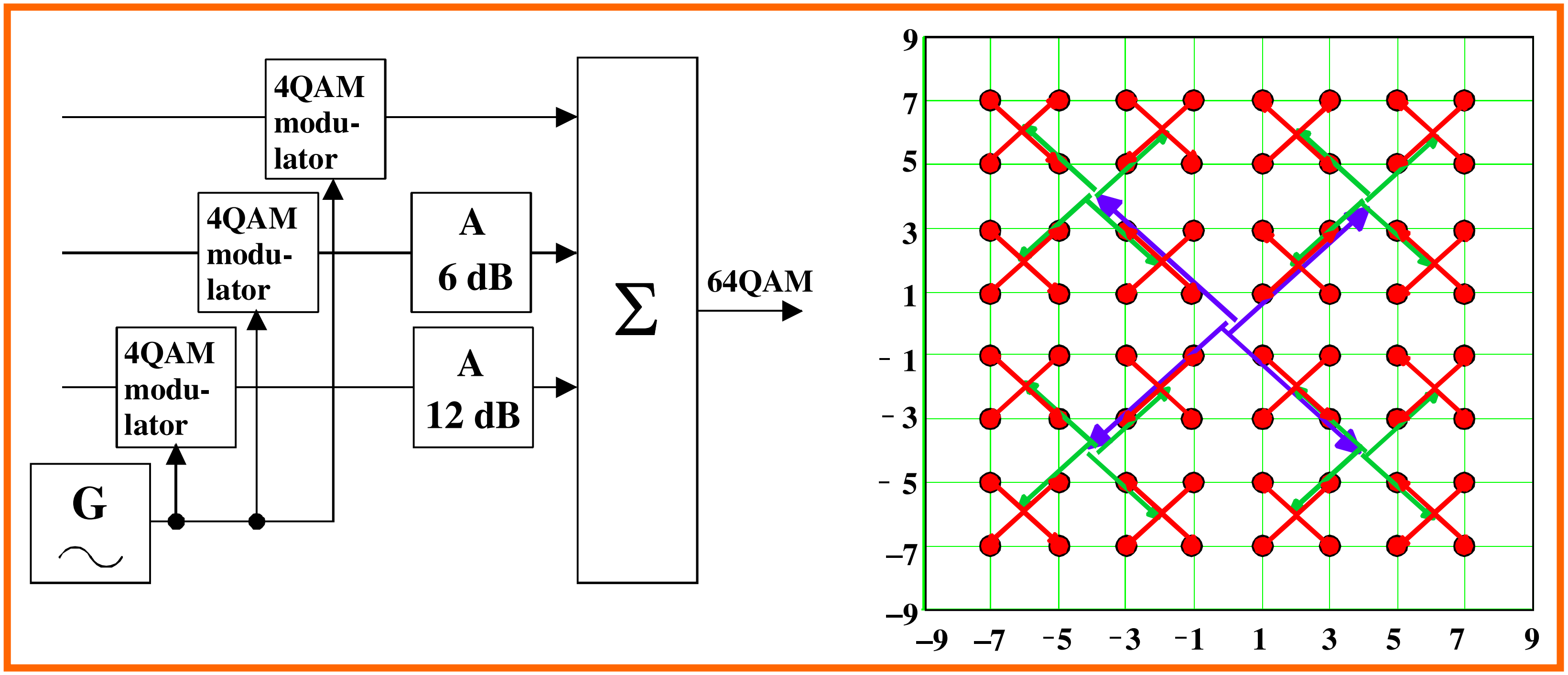
Quadrature modulation representation $t \in [kT, (k+1)T]$**:**

$$s(t) = \sqrt{2P_c} \sum_{k=-\infty}^{\infty} \left[\cos\left(\frac{\pi}{M} a_k \right) rect\left(\frac{t - 0.5T - kT}{T} \right) \cos(2\pi f_c t) - \sin\left(\frac{\pi}{M} a_k \right) rect\left(\frac{t - 0.5T - kT}{T} \right) \sin(2\pi f_c t) \right]$$

MPSK-CONSTELLATIONS



An alternative MQAM modulator



Carrier demodulator

Tasks:

- **To demodulate the modulating signal**
- **To produce signal quadrature components for coherent demodulation/data detection**
- **Carrier recovery for coherent demodulation**
- **Analog to digital conversion (ADC) of the signal components**

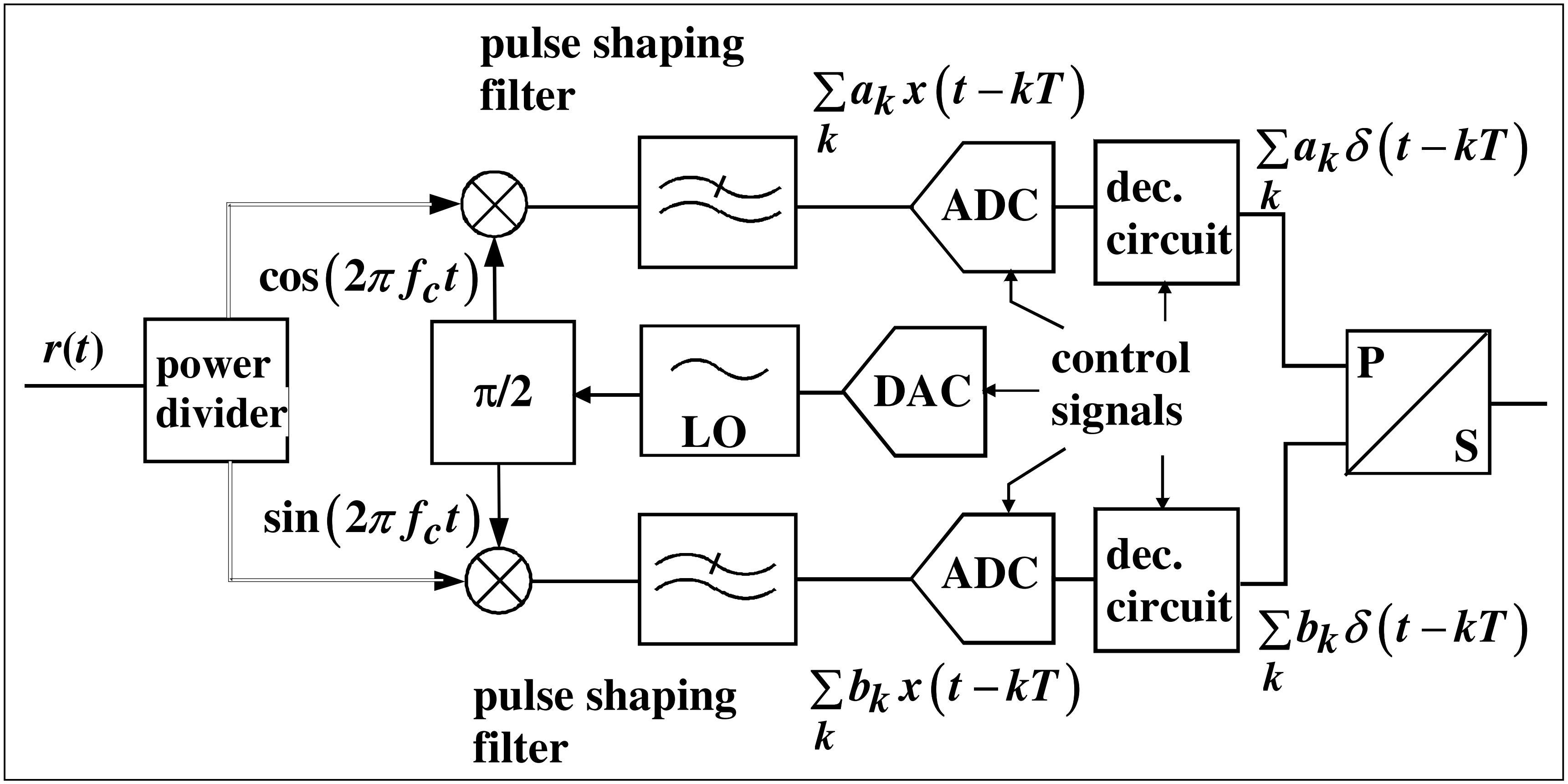
Characteristics:

- **Coherent detection is a multiplication procedure**
- **Low-pass filtering rejects the mixing result at twice the IF-frequency**
- **Sampling for ADC at symbol rate**

Problems:

- **Phase quadrature over the whole signal bandwidth**
- **Quadrature amplitude balance over the whole signal bandwidth**
- **Recovered carrier phase bias and noise**

The quadrature demodulator



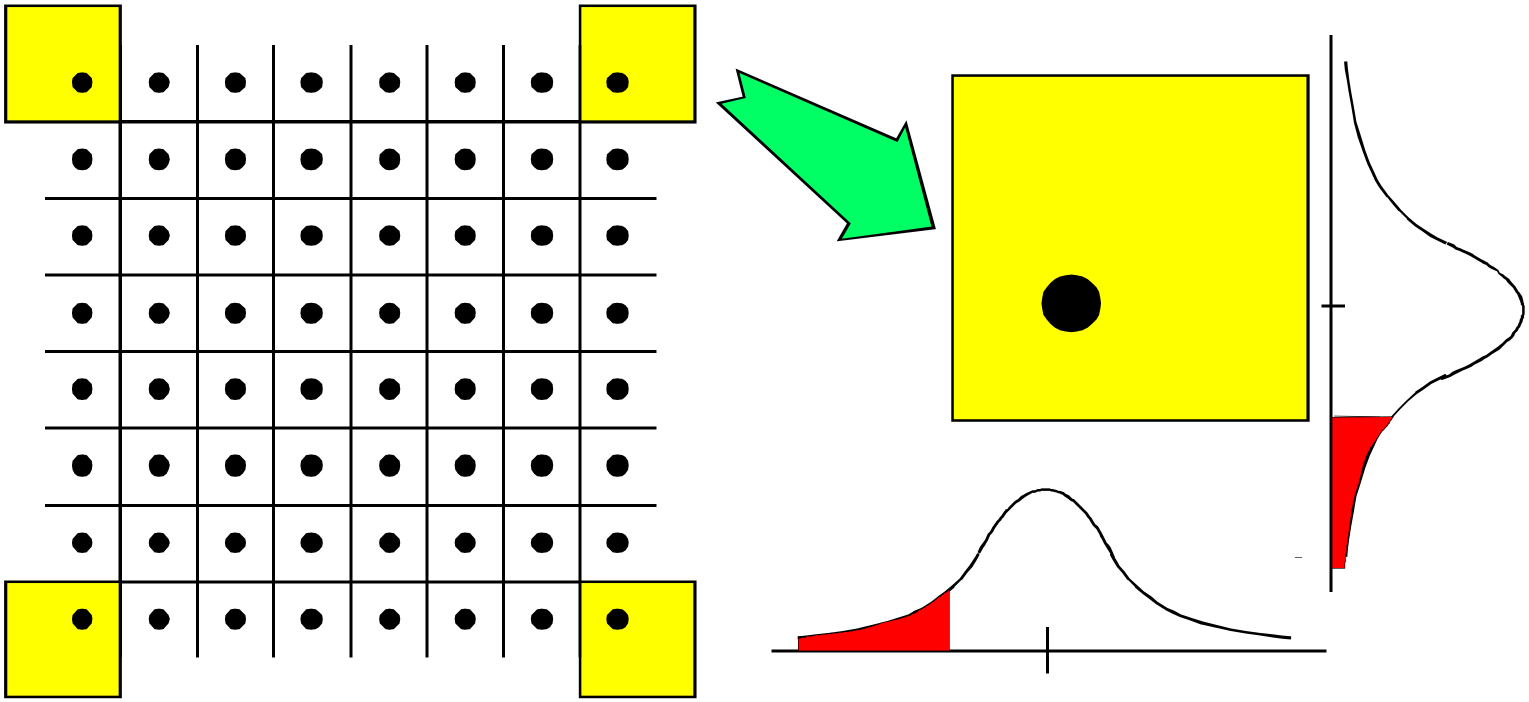
Optimum receiver bit error probability

BEP-expression are derived under certain given conditions:

- **single symbol reception or intersymbol-interference free sequence reception,**
- **The pulse waveforms used for each possible symbol are represented by minimum size orthonormal function set spanning a vector space,**
- **receiver filter(s) matched to received pulse waveforms,**
- **Maximum a posteriori signal (MAP) processing → the vector space is divided into non-overlapping decision sub-spaces for each symbol → the decision will be the symbol corresponding to the subspace where the noisy received signal vector is → minimum symbol error probability,**
- **white Gaussian noise,**
- **optimum bit combination mapping to the symbols (Gray coding)**
- **fully known received carrier phase/four-fold phase ambiguity,**
- **in case of average SEP in the Rayleigh-fading AWGN channel, the channel amplitude will not change during one symbol, but all channel states are visited during the period over which SEP is determined**

MQAM SYMBOL ERROR PROBABILITY

4 corner decision areas



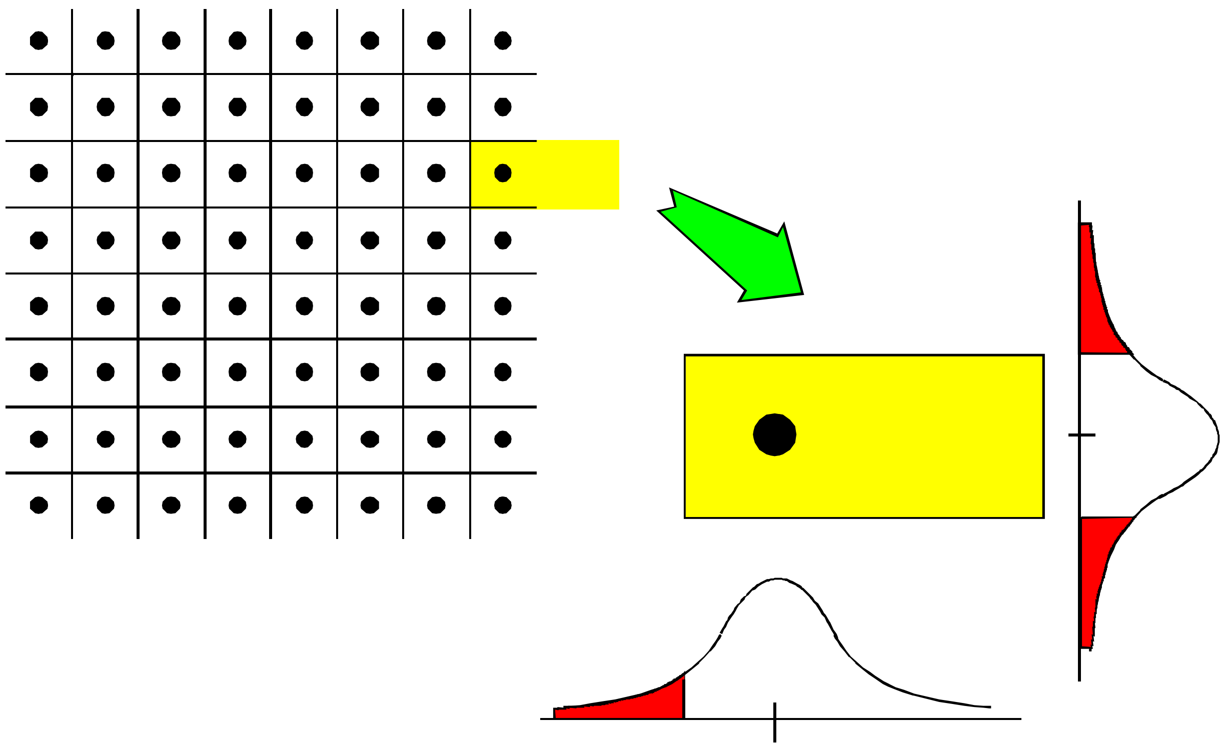
$$P\{C|m_1\} = (1 - Q(z))(1 - Q(z))$$

$$P\{E|m_1\} = 2Q(z) - Q^2(z)$$

MQAM_MPSK_MQPRS.dsf

MQAM SYMBOL ERROR PROBABILITY

$4(\sqrt{M}-2)$ edge decision areas



$$P\{C|m_2\} = (1-Q(z))(1-2Q(z))$$

$$P\{E|m_2\} = 3Q(z) - 2Q^2(z)$$

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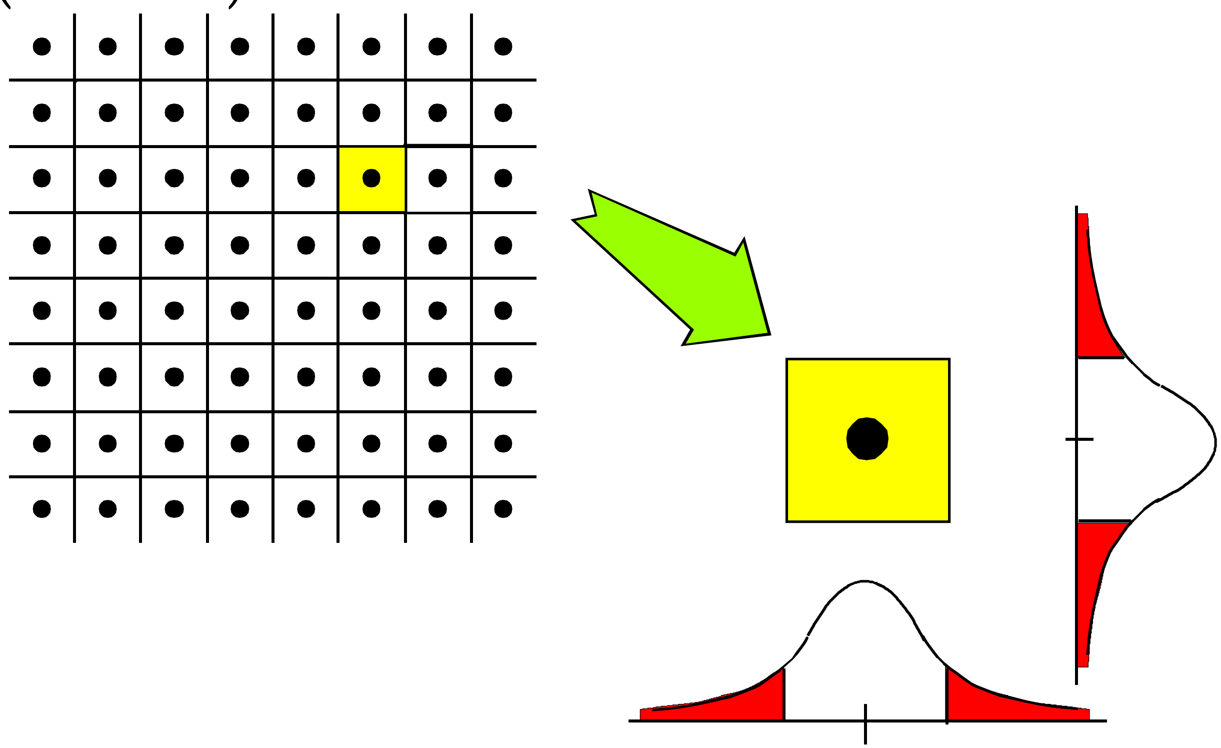
MQAM SYMBOL ERROR PROBABILITY

$\left(\sqrt{M}-2\right)^2$ **inner decision areas**



$$P\{C|m_3\} = (1-2Q(z))(1-2Q(z))$$

$$P\{E|m_3\} = 4Q(z) - 4Q^2(z)$$

MQAM_MPSK_MQPRS.dsf

Average symbol error probability

$$P_s\{E\} = \sum_{k=1}^{M} P\{m_k\}P\{E|m_k\} = \frac{1}{M}\sum_{k=1}^{M} P\{E|m_k\}$$

$$= \frac{1}{M}\left[N_{m_1}P\{E|m_1\} + N_{m_2}P\{E|m_2\} + N_{m_3}P\{E|m_3\}\right]$$

$$= \frac{1}{M}\left[4\left(2\mathrm{Q}(z) - Q^2(z)\right) + 4\left(\sqrt{M}-2\right)\left(3\mathrm{Q}(z) - 2Q^2(z)\right) + \right.$$
$$\left. + \left(\sqrt{M}-2\right)^2\left(4\mathrm{Q}(z) - 4Q^2(z)\right)\right]$$

$$= \frac{1}{M}\left[\left(8 + 12\sqrt{M} - 24 + 4M - 16\sqrt{M} + 16\right)\mathrm{Q}(z) + \right.$$
$$\left. + \left(-4 - 8\sqrt{M} + 16 - 4M + 16\sqrt{M} - 16\right)\mathrm{Q}^2(z)\right]$$

$$= \frac{1}{M}\left[\left(4M - 4\sqrt{M}\right)Q(z) - \left(4M - 8\sqrt{M} + 4\right)Q^2(z)\right]$$

$$= \frac{4}{M}\left[\left(M - \sqrt{M}\right)Q\left(\frac{d}{\sigma_n}\right) - \left(\sqrt{M} - 1\right)^2 Q^2\left(\frac{d}{\sigma_n}\right)\right]$$

With an optimum receiver in the AWGN-channel the symbol error probability is given by the expression

$$P_s\{E\} = 4\left(1 - \frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{2E_o}{N_o}}\right) - 4\left(1 - \frac{1}{\sqrt{M}}\right)^2 Q^2\left(\sqrt{\frac{2E_o}{N_o}}\right)$$

E_o is the energy of any of the 4 innermost in-phase/quadrature pulse shapes

Average and peak energy in a MQAM-system

Peak energy
(energy of a corner symbol)

$$E_h = |s_i|_{\max} = \left(\sqrt{M} - 1\right)^2 E_o$$

Average energy

$$\bar{E}_s = \frac{1}{M}\sum_{i=1}^{M} |s_i|^2 = \frac{(M-1)}{3} E_o$$

	$\bar{E}_s$	E_h	$10\lg\left(\frac{\bar{E}_s}{E_o}\right)$	$10\lg\left(\frac{E_h}{\bar{E}_s}\right)$
4QAM	E_o	E_o	0dB	0dB
16QAM	$5E_o$	$9E_o$	7.0	1.5
64QAM	$21E_o$	$49E_o$	13.2	3.7
256QAM	$85E_o$	$225E_o$	19.3	4.2
1024QAM	$341E_o$	$961E_o$	25.3	4.5

MQAM_MPSK_MQPRS.dsf

QAM-error probability in the LTI-AWGN-channel

With matched filtering and a fully known carrier phase and ideal symbol timing the symbol error probability of general quadratic QAM is

$$P_s = 4\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3}{M-1}\cdot\gamma}\right)-4\left(1-\frac{1}{\sqrt{M}}\right)^2 Q^2\left(\sqrt{\frac{3}{M-1}\cdot\gamma}\right)$$

where

$$\gamma = \frac{P_{rx}}{N_o R_s}$$

With Gray-coding (giving a minimum of different bits in adjacent symbols) the bit error probability (BEP) in 4QAM and 16QAM is:

$$P_{b,4QAM} = Q\left(\sqrt{\gamma}\right)$$

$$P_{b,16QAM} = 0.75Q\left(\sqrt{0.2\gamma}\right)+0.5Q\left(\sqrt{1.8\gamma}\right)-0.25Q\left(\sqrt{12.5\gamma}\right)$$

Differentially encoded MQAM

If an MQAM-signal is generated by adding 4QAM-signals, and the different 4QAM-signals are separately demodulated, then if the absolute carrier phase is unknown, all signals have to be separately differentially encoded, and the symbol and bit error probabilities are doubled..

If the in-phase and quadrature √MPAM-signals are demodulated, the differential encoding must be done in another way, and according to Weber [x] the bit error probability is not doubled and it is obtained with the following expression

$$P_{b,de} = \left[1 + \frac{\log_2 M}{2\left(\sqrt{M} - 1\right)}\right] P_b =$$

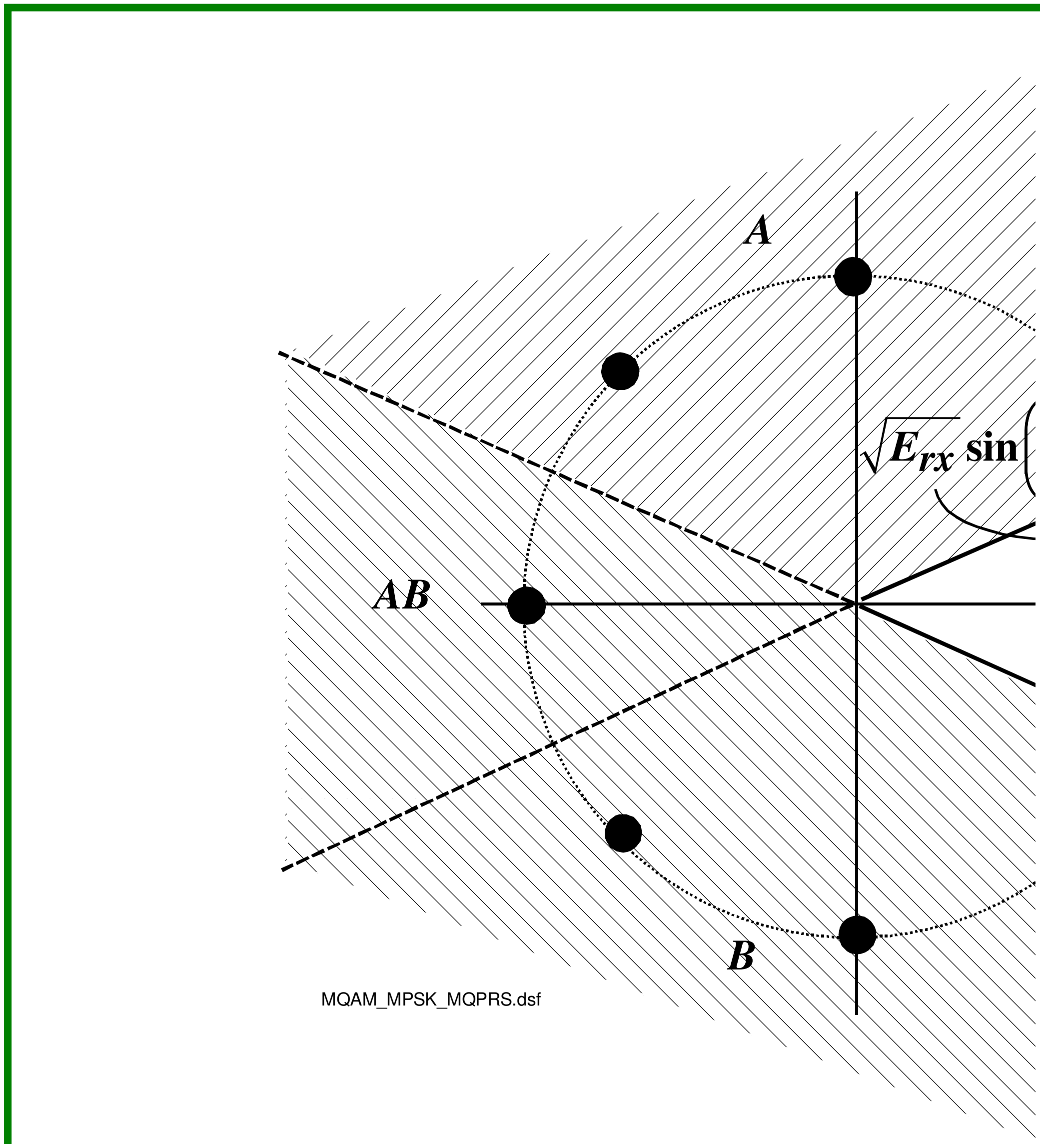
$$\cong \left[1 + \frac{\log_2 M}{2\left(\sqrt{M} - 1\right)}\right] \frac{4}{\log_2 M} \left[1 - \frac{1}{\sqrt{M}}\right] \mathrm{Q}\left(\sqrt{\frac{3}{M-1} \cdot \frac{x^2(0)}{\sigma_n^2}}\right)$$

Error probability in MPSK

Error event: Due to noise the received vector is in the shaded area

An exact expression for the symbol error probability is not easily obtained, only an upper bound can be given.

We define two sub-events:
A: the signal vector is in the upper half-plane
B: the signal vector is in the lower half-plane



The probability that the vector is in either half-plane is easily obtained, it is given by

$$P\left\{e \middle| m_1\right\} = \mathrm{Q}\left(\frac{d}{\sigma_n}\right) = \mathrm{Q}\left(\sqrt{\frac{E_{rx}\sin^2(\pi/M)}{N_o/2}}\right) = \mathrm{Q}\left(\sqrt{\frac{2E_{rx}}{N_o}}\sin\left(\frac{\pi}{M}\right)\right)$$

The latter expressions are valid for a matched filter receiver with known carrier phase

The average symbol error probability is upper bounded in the following way

$$\begin{aligned} P_s(e|m_1) &= P_s(e) = P\left\{r_1 \in A + B\right\} \\ &= P\left\{r_1 \in A\right\} + P\left\{r_1 \in B\right\} - P\left\{r_1 \in AB\right\} \\ &\le P\left\{r_1 \in A\right\} + P\left\{r_1 \in B\right\} = 2\mathrm{Q}\left(\sqrt{\frac{2E_{rx}}{N_o}}\sin\left(\frac{\pi}{M}\right)\right) \end{aligned}$$

The first equality is due to the symmetry of the constellation

The bit error probability is upper bounded by

$$P_b(e) \le \frac{P_s(e)}{\log_2 M} = \frac{2}{\log_2 M} Q\left(\sqrt{\frac{2E_{rx}}{N_o}} \sin\left(\frac{\pi}{M}\right)\right)$$

$$= \frac{2}{\log_2 M} Q\left(\sqrt{\frac{2E_b \log_2(M)}{N_o}} \sin\left(\frac{\pi}{M}\right)\right)$$

If the carrier phase is unknown or ambiguous differential encoding must be applied.
As the symbol error will occur in pairs, the error probability is doubled:

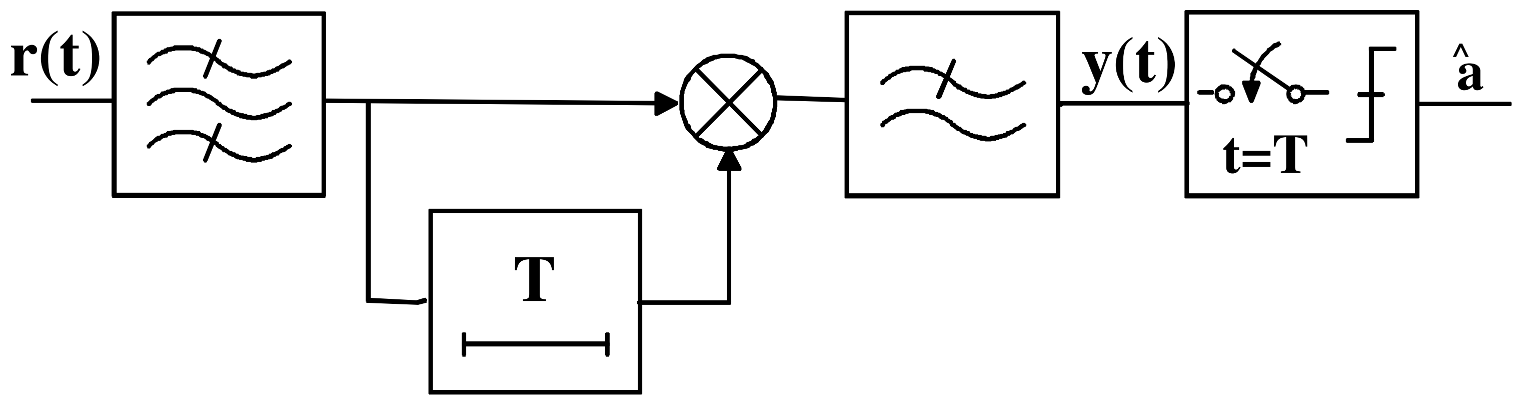
$$P_{b,de}(e) \le \frac{4}{\log_2 M} Q\left(\sqrt{\frac{2E_{rx}}{N_o}} \sin\left(\frac{\pi}{M}\right)\right)$$

$$= \frac{4}{\log_2 M} Q\left(\sqrt{\frac{2E_b \log_2(M)}{N_o}} \sin\left(\frac{\pi}{M}\right)\right)$$

Non-coherent detection of PSK (DPSK)

- Principle of non-coherent detection of PSK-signals:
  The received signal delayed by one symbol interval is used as a coherent local carrier

- Block diagram of the receiver



The delay circuit makes the change of symbol rate rather difficult

- Function of the demodulator/

  The received signal is

$$r(t) = \sum_{k=-\infty}^{\infty} x(t-kT)\cos(2\pi f_c t + a_k\pi + \phi_o) + w(t)$$

$$= x(t-kT)\cos(2\pi f_c t + a_k\pi + \phi_o)$$

$$+ \sum_{\substack{l=-\infty \\ l\neq k}}^{\infty} x(t-kT)\cos(2\pi f_c t + a_k\pi + \phi_o) + w(t)$$

$$= x(t-kT)\cos(2\pi f_c t + a_k\pi + \phi_o) + isi_k(t) + w(t)$$

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- a_k is MPAM-data, that get the values $\pm 1, \pm 3, \ldots, \pm(M-1)$
- ϕ_o is the basic carrier phase,
- $w(t) = n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$ is zero mean white Gaussian noise
- f_c is the carrier frequency
- isi_k is the instantaneous inter-symbol interference
- $T = 1/R_s$ is the symbol duration

The received signal after the one symbol delay is

$$r(t-T)=\sum_{k=-\infty}^{\infty} x(t-kT)\cos\left(2\pi f_c(t-T)+a_{k-1}\pi+\phi_o\right)+w(t-T)$$

$$=x(t-kT)\cos\left(2\pi f_c t+a_{k-1}\pi+\phi_o\right)$$

$$+\sum_{\substack{l=-\infty \\ l\neq k}}^{\infty} x(t-kT)\cos\left(2\pi f_c t+a_{k-1}\pi+\phi_o\right)+w(t-T)$$

$$=x(t-kT)\cos\left(2\pi f_c t+a_{k-1}\pi+\phi_o\right)+isi_{k-1}(t)+w(t-T)$$

After the multiplier the product is

$$r(t)r(t-T)=\left[x(t-kT)\cos\left(2\pi f_c t+a_k\pi+\phi_o\right)+isi_k(t)+w(t)\right]$$

$$\cdot\left[x(t-kT)\cos\left(2\pi f_c t+a_{k-1}\pi+\phi_o\right)+isi_{k-1}(t)+w(t-T)\right]$$

Omitting the ISI and noise the signal part after low-pass filtering is

$$y(t)=P_{rx}\cos\left(2\pi f_c T+(a_k-a_{k-1})\pi\right)\cdot x^2\left(\frac{t-kT}{T}\right)$$

If $2\pi f_c T$ is an integer the output signal is

$$y(t) = P_{rx} \cos\left((a_k - a_{k-1})\pi\right) x^2\left(\frac{t-kT}{T}\right)$$

Assuming decision sampling at the time instant of the normalized maximum of the squared transmitted waveform the following values occur for a bipolar binary signal

a_k	a_{k-1}	y
–1	–1	$-2P_{rx}$
–1	+1	0
+1	–1	0
+1	+1	$+P_{rx}$

- If the input data changes from one symbol interval to the next: $y = 0$
  The input data does not change: $y \neq 0$

By treating the QPSK-signal as two independent BPSK-signals in quadrature we can obtain the same behavior

- As the changes can be directly decided, it follows that the input data should be correspondingly encoded.

- Differential encoding which is explained later leads to desired behaviour, no differential decoding is needed as in coherent demodulation

Error performance of differentially demodulated PSK

The derivation is rather complex, especially for QPSK

DBPSK, optimum receiver

$$P_b = 0.5 \exp(-\gamma)$$

DMPSK, optimum receiver

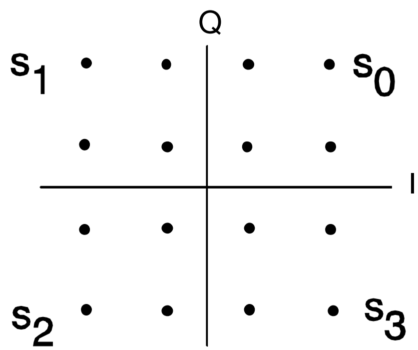
$$P_b \cong \frac{1}{\log_2 M}\left[1 - \mathrm{Q}\left(\sqrt{\gamma\left(1+\sin\frac{\pi}{M}\right)}, \sqrt{\gamma\left(1-\sin\frac{\pi}{M}\right)}\right) + \mathrm{Q}\left(\sqrt{\gamma\left(1-\sin\frac{\pi}{M}\right)}, \sqrt{\gamma\left(1+\sin\frac{\pi}{M}\right)}\right)\right]$$

Q(x,y) is Marcum's Q-function which is defined by the expression

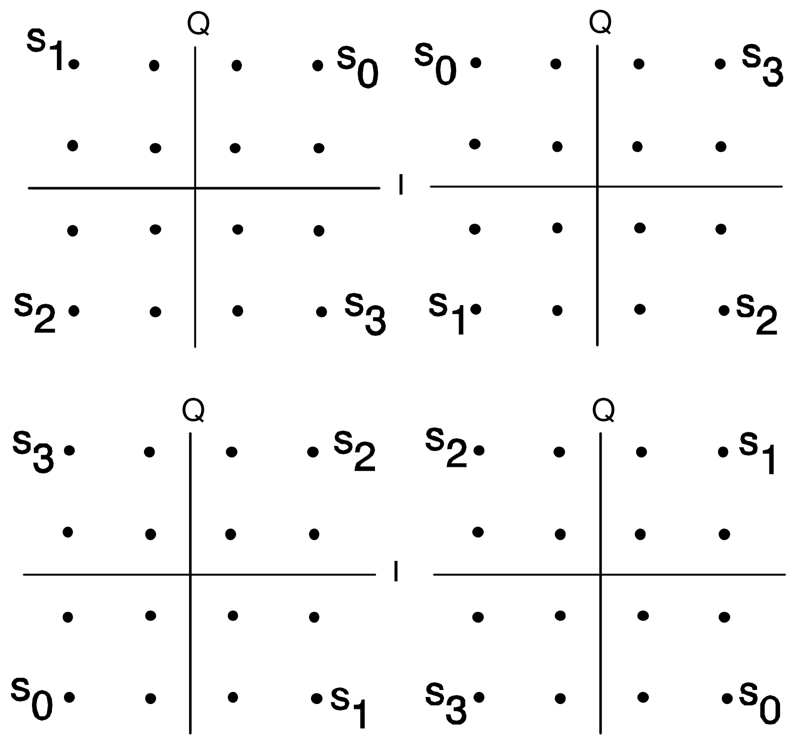
$$\mathrm{Q}(x, y) = \exp\left(-\frac{x^2 + y^2}{2}\right) \sum_{k=0}^{\infty} \left(\frac{x}{y}\right)^k \mathrm{I}_k(xy)$$

Phase ambiguity in 16QAM

Transmitted constellation



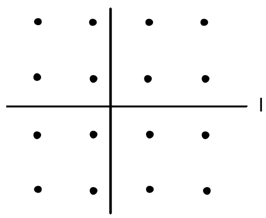
Possible received constellations with phase unambiguuity



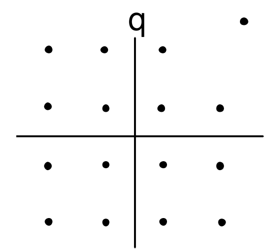
Methods for removal of phase ambiguity

Adding a pilot signal

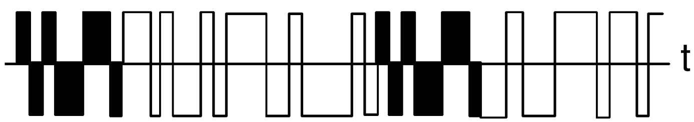
$s(t)=s_0(t)+a\cos(2\pi f_c t)$



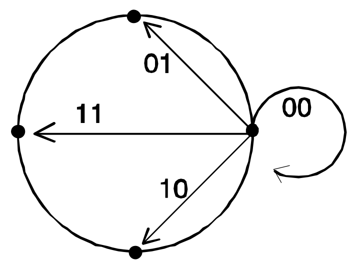
Changing one constellation point



Adding a learning sequence



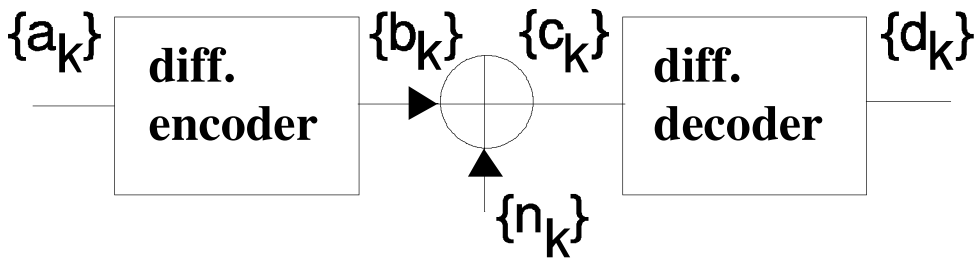
Differential encoding and decoding



symbol	phase change
00	0
01	$\pi/2$
11	π
10	$-\pi/2$

DIFFERENTIALLY ENCODED MPSK

Differential encoder and decoder



k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a_k		2	3	1	2	0	2	1	3	3	3	3	2	0	0	1	2	1	0	0	1
b_{k-1}		0	2	1	2	0	0	2	3	2	1	0	3	1	1	1	2	0	1	1	1
b_k	0	2	1	2	0	0	2	3	2	1	0	3	1	1	1	2	0	1	1	1	2

phase state 1

c_k	0	2	1	2	0	0	2	3	2	1	0	3	1	1	1	2	0	1	1	1	2
d_k	0	2	3	1	2	0	2	1	3	3	3	3	2	0	0	1	2	1	0	0	1

phase state 2

c_k	0	3	2	3	1	1	3	0	3	2	1	0	2	2	2	3	1	2	2	2	3
d_k	0	3	3	1	2	0	2	1	3	3	3	3	2	0	0	1	2	1	0	0	1

phase state 3

c_k	0	0	3	0	2	2	0	1	0	3	2	1	3	3	3	0	2	3	3	3	0
d_k	0	0	3	1	2	0	2	1	3	3	3	3	2	0	0	1	2	1	0	0	1

phase state 4

c_k	0	1	0	1	3	3	1	2	1	0	3	2	0	0	0	1	3	0	0	0	1
d_k	0	1	3	1	2	0	2	1	3	3	3	3	2	0	0	1	2	1	0	0	1

phase state 4 with symbol decision errors

c_k	0	1	0	1	2	3	1	2	1	0	1	2	0	2	1	1	3	0	1	0	1
d_k	0	1	3	1	1	2	2	1	3	3	1	1	2	2	3	0	2	1	1	3	1

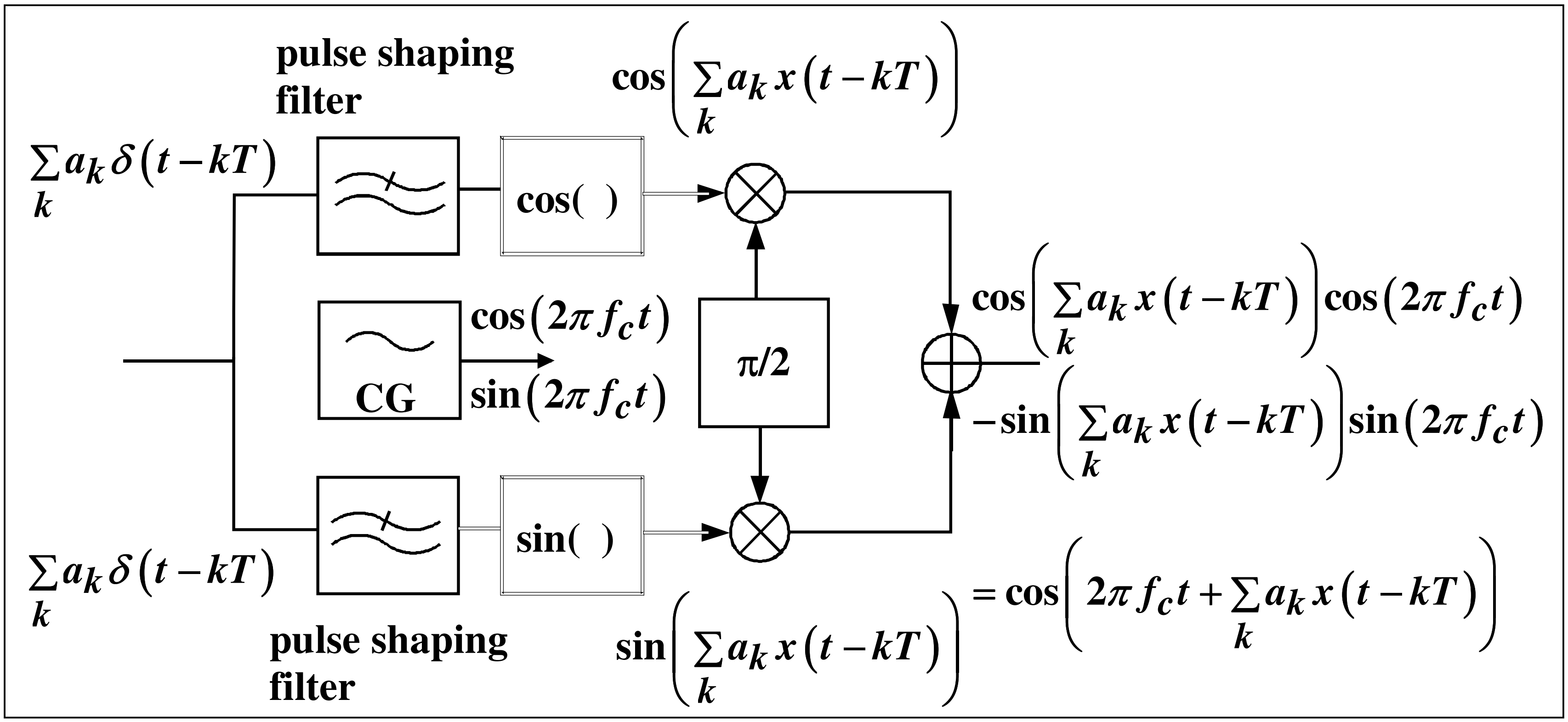
single error $\rightarrow$ double errors

subsequent errors $\rightarrow$ one additional error

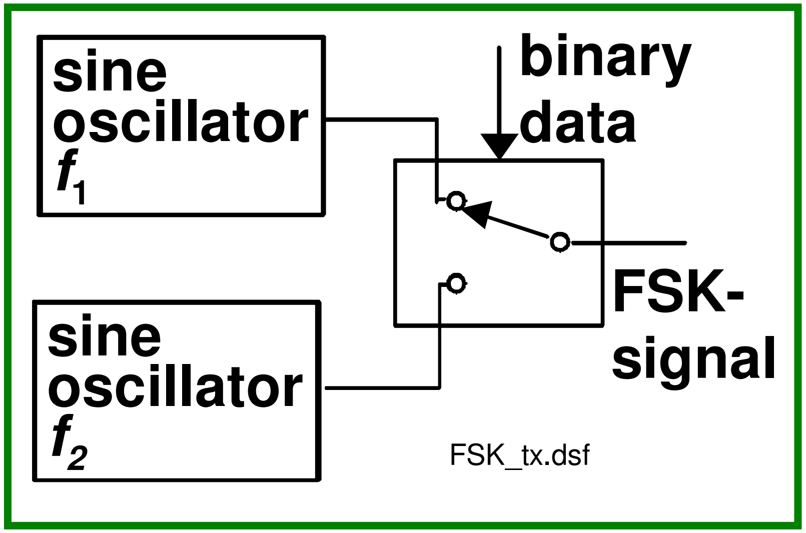
$$P_{s,MDPSK}(E) \cong 2P_{s,MPSK}(E) \qquad P_{b,MDPSK}(E) \cong 2P_{b,MPSK}(E)$$

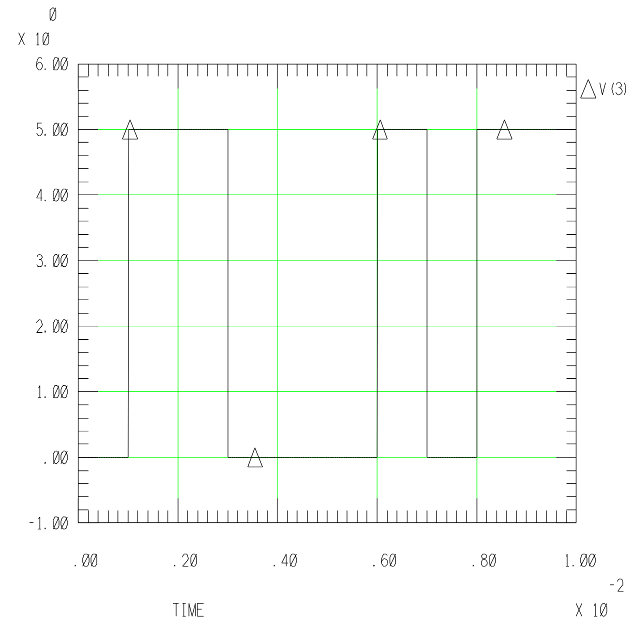
Non-linear modulation methods

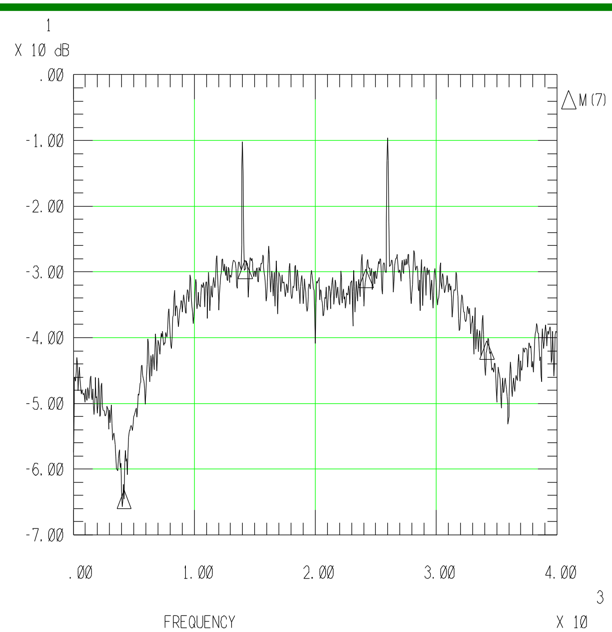
Quadrature modulator as non-linear modulator

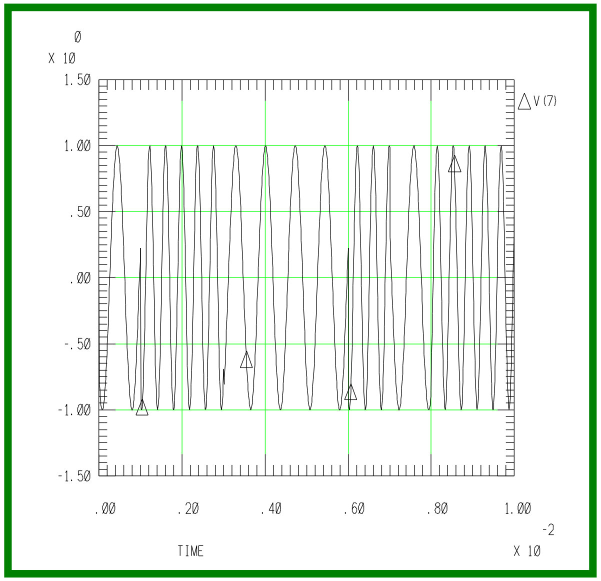


FSK

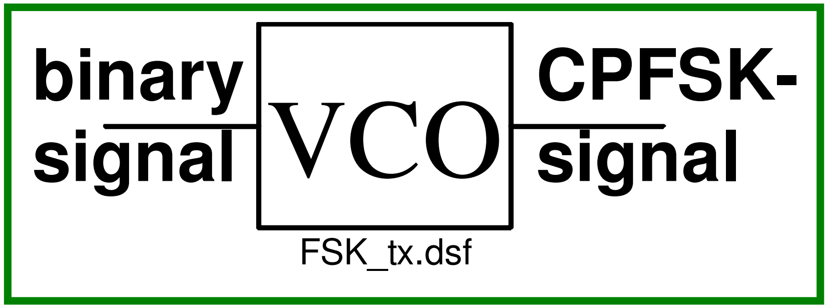

sine oscillator f_1
binary data
sine oscillator f_2
FSK-signal
FSK_tx.dsf

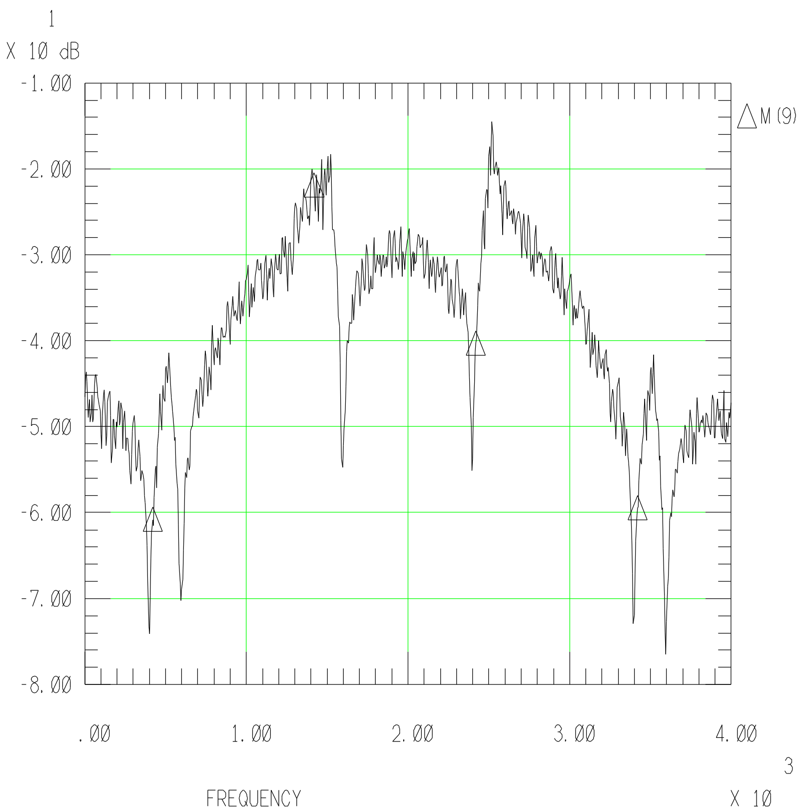

V (3)
TIME

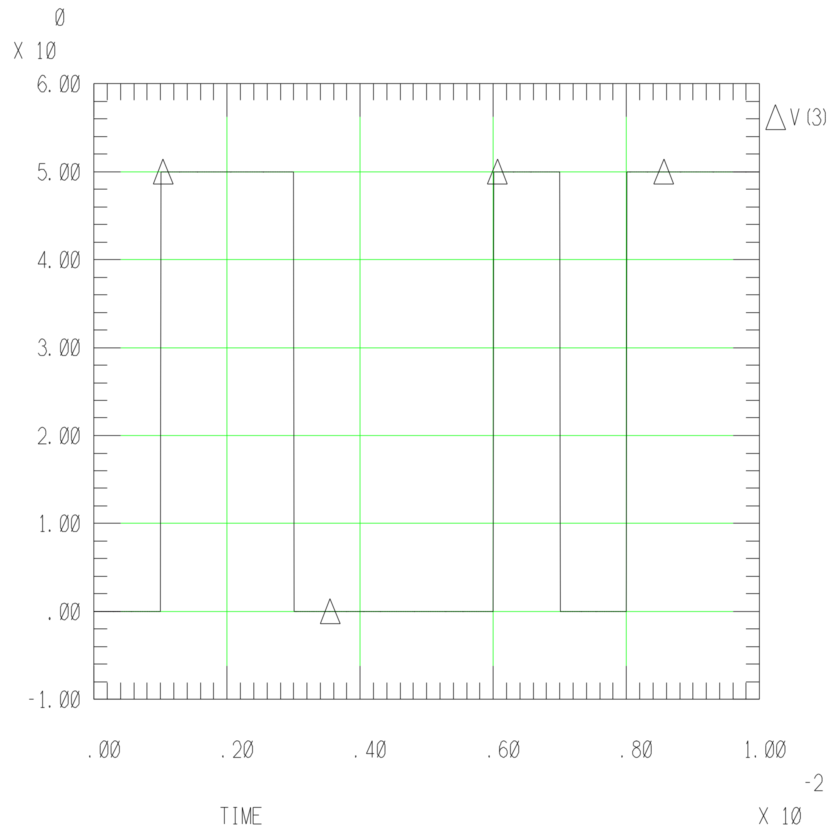

X 10 dB
M (7)
FREQUENCY

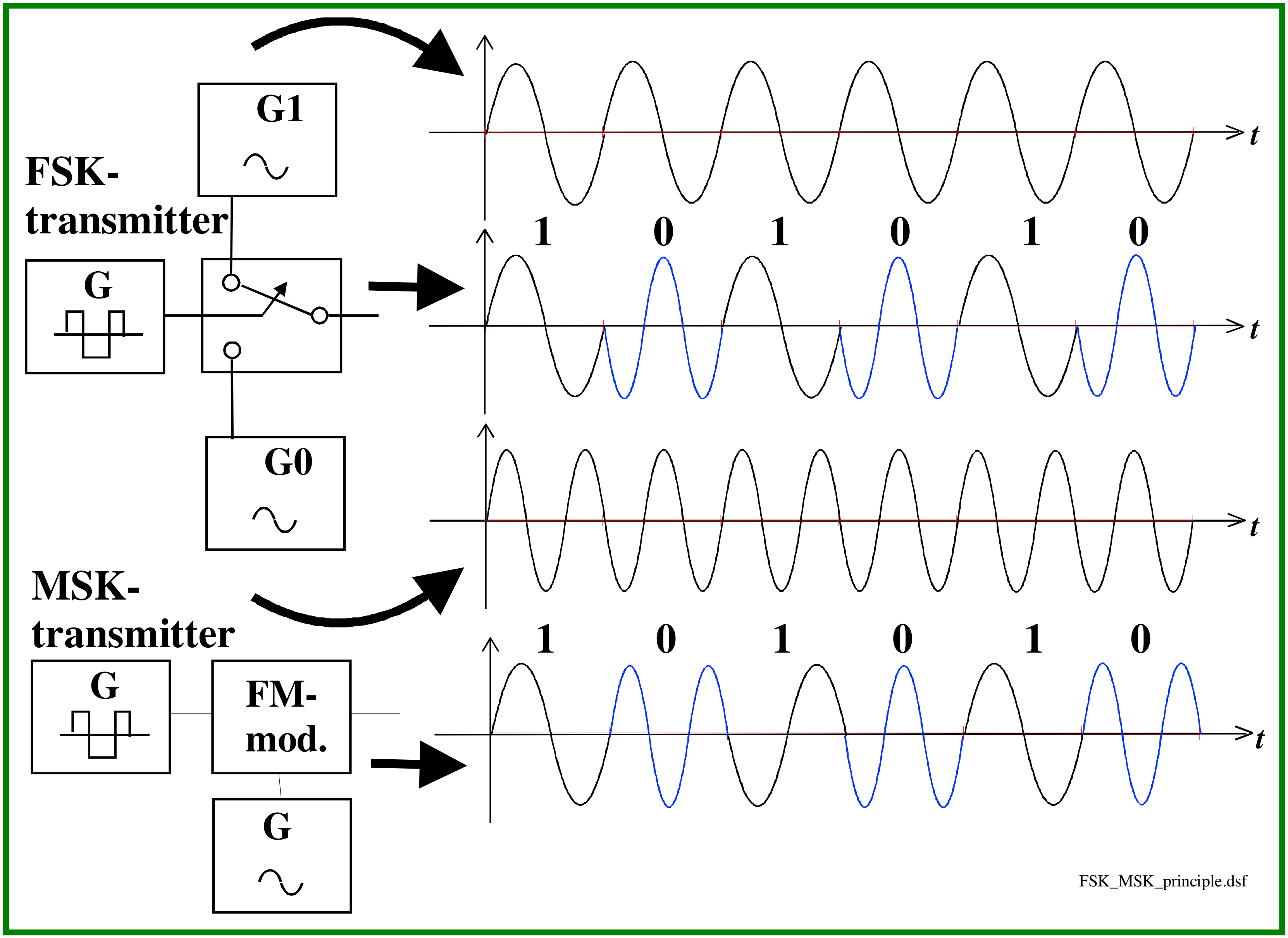

V (7)
TIME

CPFSK


binary signal
VCO
CPFSK-signal
FSK_tx.dsf

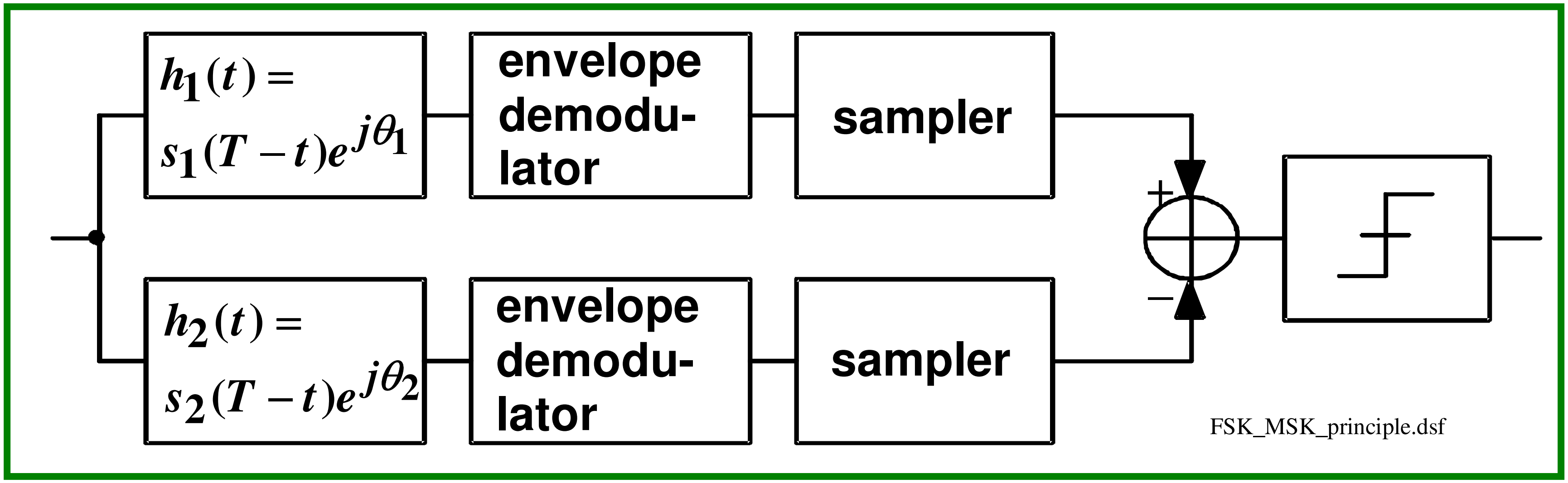

X 10^1 dB
M (9)
FREQUENCY
X 10^3


X 10^0
V (3)
TIME
X 10^-2


FSK-
transmitter
G1
G
G0
MSK-
transmitter
G
FM-
mod.
G
t
1 0 1 0 1 0
t
t
1 0 1 0 1 0
t
FSK_MSK_principle.dsf

NON-COHERENT FSK-RECEIVER

Block diagram of a non-coherent FSK-receiver



- In envelope demodulation the matched filters may have arbitrary phases
- Each data symbol generates in principles a signal only in one branch
- The data decision is made based on the sign of the sample difference

Error performance of FSK-receivers

Coherent binary FSK-receiver

$$P_b = \mathrm{Q}\left(\sqrt{\frac{E_{rx}}{N_o}}\right)$$

Non-coherent binary FSK-receiver

$$P_b = 0.5\exp\left(-\frac{E_{rx}}{2N_o}\right)$$

Coherent MSK-receiver

$$P_b = \mathrm{Q}\left(\sqrt{\frac{2E_{rx}}{N_o}}\right)$$