S-72.2205 Digital Transmission Methods

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1 Baseband channel

The baseband channel consists of a transmitter filter (Tx) that takes as input the transmitted digital symbols and gives them a waveform. Tx has different waveforms for all different symbols of the used constellation. The signal after Tx is denoted as $x(t)$. The signal experiences the communication channel $h(t)$, which is denoted by convolution $x(t) \otimes h(t)$. Then noise is added to the received signal $r(t)$. The signal model of such system reads

$$r(t) = x(t) \otimes h(t) + n(t) \tag{1}$$

At the receiver $r(t)$ is convolved with the receiver filter (Rx) with a waveform $v(t)$ to get the received signal $z(t)$ from which the decisions are made. Signal after receiver filter reads

$$z(t) = x(t) \otimes h(t) \otimes v(t) + n(t) \otimes v(t) \tag{2}$$

1.1 AWGN channel

In Additive White Gaussian Noise channel $h(t) = 1$ and the signal after receiver filter reduces to

$$z(t) = x(t) \otimes v(t) + n(t) \otimes v(t). \tag{3}$$

Assume a bipolar binary transmission system where the transmit filter generates a rectangular waveform that reads

$$x(t) = \sqrt{\frac{\mathcal{E}}{T}} rect\left(\frac{t - T/2}{T}\right) \tag{4}$$

where $\mathcal{E}$ is the energy per symbol and T is the symbol period. If Rx is matched to Tx it means that $v(t) = x(t - T)$. The signal to noise ratio (SNR) after a matched filter is $\frac{2\mathcal{E}}{N_0}$, where $N_0/2$ is the noise spectral density.

1. Calculate the SNR reduction of **single symbol transmission** when the receive filter is a sub-optimal RC-low-pass filter with the time constant λ, i.e.

$$v(t) = \frac{1}{\lambda} e^{-\frac{t}{\lambda}} u(t) \tag{5}$$

Solution

Th SNR after matched filter is known and we should figure out the SNR after this suboptimal filter. The average noise power after this receive filter is

$$\begin{aligned}\sigma_n^2 &= \int_{-\infty}^{\infty} |V(f)|^2 S_n(f) df = \frac{N_0}{2} \int_{-\infty}^{\infty} |V(f)|^2 df \\ &= \frac{N_0}{2} \int_{-\infty}^{\infty} v^2(t) dt = \frac{N_0}{2} \int_{0}^{\infty} \frac{1}{\lambda^2} e^{-\frac{2t}{\lambda}} dt \\ &= -\Big|_0^{\infty} \frac{N_0}{2} \frac{1}{2\lambda} e^{-\frac{2t}{\lambda}} = \frac{N_0}{4\lambda} \end{aligned} \tag{6}$$

The optimum error probability is obtained with the maximum signal to noise ratio thus we need to find the optimum sampling instant at the receiver. The input signal to decision making in AWGN channel without the noise part is the convolution of the transmit filter with the receive filter $y(t) = x(t) \otimes v(t)$. Three parts can be separated from the convolutional integral that reads

$$y(t) = \int_{-\infty}^{\infty} v(\mu) x(t-\mu) d\mu \tag{7}$$

First part $t < 0$ is trivial since then the filters are not yet overlapping and the result is $y(t) = 0$. Second part is $0 < t < T$ when the rect filter starts to overlapp with the receiver filter until the point $t = T$ when the filters are (first time) totally overlapping. The integral reads

$$\begin{aligned} y(t) &= \int_0^t \sqrt{\frac{\mathcal{E}}{T}} \frac{1}{\lambda} e^{-\frac{\mu}{\lambda}} d\mu = -\Big|_0^t \sqrt{\frac{\mathcal{E}}{T}} e^{-\frac{\mu}{\lambda}} \\ &= \sqrt{\frac{\mathcal{E}}{T}} \left(1 - e^{-\frac{t}{\lambda}}\right) \end{aligned} \tag{8}$$

Third part is $t > T$ and the integral reads

$$\begin{aligned} y(t) &= \int_{t-T}^t \sqrt{\frac{\mathcal{E}}{T}} \frac{1}{\lambda} e^{-\frac{\mu}{\lambda}} d\mu = -\Big|_{t-T}^t \sqrt{\frac{\mathcal{E}}{T}} e^{-\frac{\mu}{\lambda}} \\ &= \sqrt{\frac{\mathcal{E}}{T}} \left(e^{-\frac{t-T}{\lambda}} - e^{-\frac{t}{\lambda}}\right) = \sqrt{\frac{\mathcal{E}}{T}} \left(e^{-\frac{T}{\lambda}-1}\right) e^{-\frac{t}{\lambda}} \end{aligned} \tag{9}$$

The optimum sampling instant is $t_s = T$ because at that time instant the convolution has its maximum value. The signal to noise ratio at the optimum sampling instant is

$$
\begin{aligned}
SNR = \frac{y^2(t_s)}{\sigma_n^2} &= \frac{\frac{\mathcal{E}}{T}\left(1 - e^{-\frac{T}{\lambda}}\right)^2}{\frac{N_0}{4\lambda}} \\
&= \frac{2\mathcal{E}}{N_0} 2\frac{\lambda}{T}\left(1 - e^{-\frac{T}{\lambda}}\right)^2 \qquad (10)
\end{aligned}
$$

$\frac{2\mathcal{E}}{N_0}$ is the SNR of the matched filter and the remaining part tells the reduction from that. The reduction term depends on the time constant of the filter and it can be optimized by differentiation adn setting the differential to zero. By iteration the optimum value slowly converges to $\lambda = T/1.256$ and using that value the degradation is

$$
\begin{aligned}
\Delta SNR &= 10\log_1 0\frac{SNR_{MF}}{SNR_{RC}} \\
&= -10\log_1 0\left(2\frac{1}{1.256}\left(1 - e^{-1.256}\right)^2\right) = 0.89dB \qquad (11)
\end{aligned}
$$

2. Calculate the SNR reduction of **symbol sequence transmission** when the receive filter is a sub-optimal RC-low-pass filter with the time constant used in the previous task. Is the transmission in AWGN channel always free of intersymbol interference (ISI)?

Solution

If matched filter is used with the rect filter the symbol duration at the receiver is also T and there is no ISI. As seen in the previous task, after a suboptimal filter the symbol duration at the receiver is longer than T. If the symbols are transmitted consecutively the symbols are overlapping at the receiver resulting ISI. The ISI is caused by the tails of the previous symbols and the ISI term can be written as a sum of symbols that are sampled at $t_s + kt_s$

$$
ISI = \sum_{k=1}^{\infty}\sqrt{\frac{\mathcal{E}}{T}}\left(1 - e^{T/\lambda}\right)e^{\frac{t_s+kt_s}{\lambda}} = \sqrt{\frac{\mathcal{E}}{T}}\left(1 - e^{T/\lambda}\right)\sum_{k=2}^{\infty}e^{\frac{kT}{\lambda}} \qquad (12)
$$

In the last expression $t_s = T$. SNR in this case reads

$$\begin{aligned} SNR &= \frac{(y(T)-ISI)^2}{\sigma_n^2} = \frac{\frac{\mathcal{E}}{T}\left(1-e^{-\frac{T}{\lambda}}\right)^2}{\frac{N_0}{4\lambda}} \left(1-\sum_{k=2}^{\infty} e^{\frac{kT}{\lambda}}\right)^2 \\ &= \frac{2\mathcal{E}}{N_0} 2\frac{\lambda}{T}\left(1-e^{-\frac{T}{\lambda}}\right)^2 \left(1-\sum_{k=2}^{\infty} e^{-\frac{kT}{\lambda}}\right)^2 \end{aligned} \tag{13}$$

Using the time constant giving $T/\lambda = 1.256$ and taking into account four fisrt interfering terms we get the following value for the degradation for symbol sequence transmission

$$\begin{aligned} \Delta SNR &= 10\log_1 0\frac{SNR_{MF}}{SNR_{RC+ISI}} \\ &= -10\log_1 0\left(2\frac{1}{1.256}\left(1-e^{-1.256}\right)^2\left(1-\sum_{k=2}^{4} e^{-k1.256}\right)^2\right) \\ &= 1.91dB \end{aligned} \tag{14}$$

Note that the time constant is optimized for single symbol transmission. For symbol sequence transmission it can be optimized to be $\lambda = 2.48$ and the SNR degradation in this case is 1.76dB.

3. Calculate the bit error probability (BEP) degradation due to sub-optimal receive filter both in single symbol and symbol sequence transmission.

Solution

For most modulations the error probability given a certain SNR is not available in closed form and an error function (Q-function) is used. For BPSK the BER is given

$$P_b(\frac{\mathcal{E}}{N_0}) = Q\left(\sqrt{\frac{2\mathcal{E}}{N_0}}\right) = \frac{1}{2}erfc\left(sqrt\frac{\mathcal{E}}{N_0}\right) \tag{15}$$

erfc is the cumulative error function implemented in e.g. Matlab. For differential BPSK the error probability given a certain SNR is available in closed form and is given by

$$P_b(\frac{\mathcal{E}}{N_0}) = \frac{1}{2}e^{-\frac{\mathcal{E}}{N_0}} \tag{16}$$

Using the results of the previous tasks the BER curves for the three cases can be plotted. Figure 1 shows the BER curves for matched filter (MF), RC filter and RC filter with ISI when BPSK modulation is used.

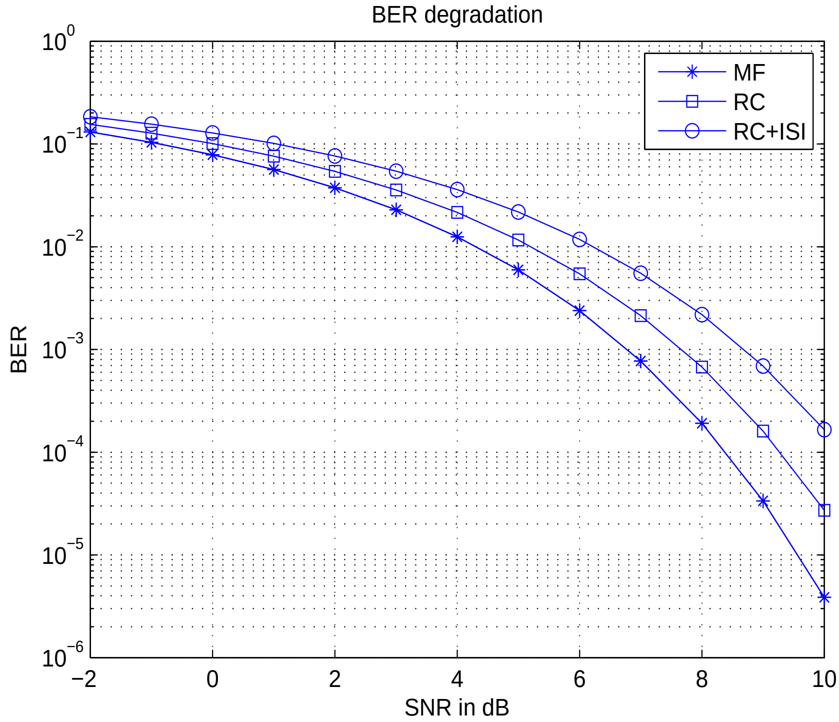

Figure 1: BER curves for matched filter (MF), RC filter and RC filter with ISI when BPSK modulation is used

1.2 Time dispersive channel

A discrete time dispersive baseband channel can be described by a **time-variant impulse response**

$$c(\tau;t) = \sum_n \alpha_n(t)\exp(-j2\pi f_c\tau_n(t))\delta[\tau - \tau_n(t)] \tag{17}$$

Which corresponds to a scenario where several (n) rays, or signal paths, intersect at the receiver. Each signal path can be described by a vector (phasor) having amplitude $\alpha_n(t)$ and phase $j2\pi f_c\tau_n(t)$, where f_c is the carrier frequency and $\tau_n(t)$ is the delay associated to the path.

There are two characteristics of the channel; τ which is the (total) time spread introduced in the signal that is transmitted through the channel and the second one is due to the time variations (in t) in the structure of the channel, that is in $\alpha_n(t)$ and $\tau_n(t)$.

1.2.1 Rayleigh fading

When n is large enough central limit theorem can be applied and $c(\tau;t)$ can be modelled as complex-valued Gaussian random process in the variable t. If we assume that all scatterers are random it makes $c(\tau;t)$ zero-mean complex Gaussian and the envelope $|c(\tau;t)|$ Rayleigh-distributed at any instant t. The channel is said to be Rayleigh fading. When α is Rayleigh fading, α^2 has chi-square probability distribution with two degrees of freedom.

1. At which average SNR value the instantaneous SNR does not fall below a threshold -10dB for 99% of time?

   Solution

   The instantaneous SNR reads $\gamma = \alpha^2\mathcal{E}/N_0$ and because α is chi-square-distributed so is γ

   $$p(\gamma) = \frac{1}{\bar{\gamma}}e^{-\gamma/\bar{\gamma}} \tag{18}$$

   where $\bar{\gamma} = \mathcal{E}/N_0 E(\alpha^2)$ is the average SNR. Figure 2 shows an example of the instantaneous received SNR behaviour in Rayleigh fading channel. The task is to tune the average SNR such that the instantaneous SNR is not below -10dB more than 1% of time. In that figure it would refer to 0.1 second.

   A cumulative distribution function (CDF) of the SNR $F(\gamma) = P(\gamma' < \gamma)$ answers to the question of the probability of an SNR value being below a threshold value. CDF is an integral of the probability distribution function of the SNR from 0 to γ

   $$F(\gamma) = \int_0^\gamma \frac{1}{\bar{\gamma}}e^{-\gamma/\bar{\gamma}}d\gamma = \Big|_0^\gamma - e^{-\gamma/\bar{\gamma}} = 1 - e^{-\gamma/\bar{\gamma}} \tag{19}$$

   Use the CDF to answer the question with the given numbers

   $$F(10) = 1 - e^{-10/\bar{\gamma}} = 0.01 \tag{20}$$

   It follows

   $$\bar{\gamma} = \frac{10}{-ln0.99} = 995 \rightarrow 29.98dB \tag{21}$$

2. For which % of time the fade depth is above 20dB? Fade depth describes how much the instantaneous SNR goes below the average SNR i.e. $10log\gamma/\bar{\gamma}$.

   Solution

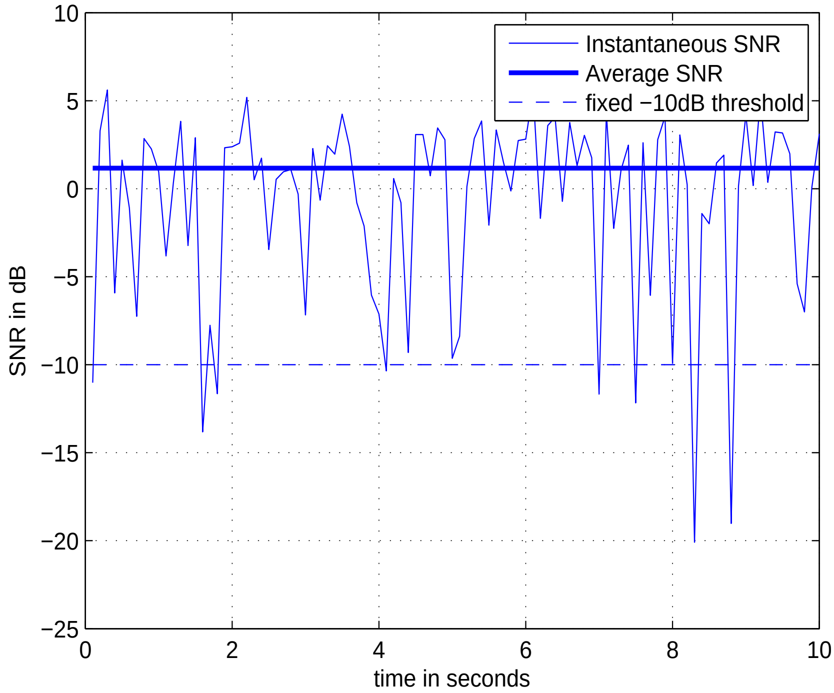

Figure 2: instantaneous SNR behaviour in Rayleigh fading channel in dB

The CDF directly anwers to that question we only need to find the absolute value of the fade depth $10log\gamma/\bar{\gamma} = 20dB$ then $\gamma/\bar{\gamma} = 10^{-2}$

$$F(\gamma) = 1 - e^{-10^{-2}} = 0.01 \tag{22}$$

Note that in this task we did not have fixed threshold but a relative value that depends on the average SNR.

3. Give the expression for the average bit error probability of the BPSK signal.

Solution

Because the SNR is not constant but varying the equation (15) gives only the instantaneous value for BER. To get the average bit error probability we need to integrate over all possible SNR values taking into account their occurrence probability. We need to calculate

$$P_b = \int_0^\infty p(\gamma) P_b(\gamma) d\gamma \tag{23}$$

The expression for BPSK in Rayleigh fading channel is

$$P_b = \int_0^\infty \frac{1}{\bar{\gamma}} e^{-\gamma/\bar{\gamma}} Q\left(\sqrt{2\gamma}\right) d\gamma \tag{24}$$

1.2.2 Parameters that characterize time dispersive channel

To define the characteristics due to τ and variation in t of the time dispersive channel the channel correlation functions are used. To simplify these functions two assumptions, wide–sense stationarity and uncorrelated scattering (WSSUS), are made. Define the autocorrelation function of the channel

$$\frac{1}{2} E[c^*(\tau_1; t) c(\tau_2; t + \Delta t)] = \phi(\tau_1; \Delta t) \delta(\tau_1 - \tau_2) \tag{25}$$

Other correlation functions are obtained by the Fourier transforms of the autocorrelation function as described in lectures. From these functions we may derive different parameters to describe the channel, i.e. **delay spread** (also called **maximum excess delay**) T_m and **coherence bandwidth** $(\Delta f)_c$, **Doppler spread** B_d and **coherence time** $(\Delta t)_c$. The variable τ is in principle responsible for $(\Delta f)_c$ and T_m that describe the frequency selectivity of the channel whereas the time selectivity, that is, how fast the channel changes in t is described by $(\Delta t)_c$ and B_d.

1. Calculate the Doppler bandwidth in the GSM900 system, when the mobile station speed is 3, 120 and 500 km/h when only reflections from stationary structures are considered. Is the channel fast or slowly fading in these cases?

   Solution

   The constant speed causes a constant Doppler-shift υ (Upsilon)

   $$\upsilon = \frac{v}{c} f_c \cos(\beta) \tag{26}$$

   where v on speed of the mobile station, c is the propagation speed of the radio wave, f_c is the carrier frequency, and β is the angle between the propagation path and the mobile station velocity vector. The Doppler bandwidth is between υ and $-\upsilon$

   $$B_d = \frac{2v}{c} f_c \tag{27}$$

   When $v = 3km/h$, $f_c = 960MHz$

   $$\tag{28}$$

Table 1:

Vehicle speed	3 km/h	120 km/h	500 km/h
GSM $T_s = 5\mu s$	$B_d = 5.33Hz$ $(\Delta t)_c = 0.19s$ S	$B_d = 213.33Hz$ $(\Delta t)_c = 4.7\mu s$	$B_d = 888.89Hz$ $(\Delta t)_c = 1.1\mu s$

The Doppler bandwidth for the other speeds are obtained in the same way. The system bandwidht is 200kHz for GSM and the corresponding symbol duration T_s is $5\mu s$. The channel is slowly fading if $T_s \ll (\Delta t)_c$. The coherence time can be approximated from the Doppler spread $(\Delta t)_c \approx 1/B_d$. Table **??** show B_d and $(\Delta t)_c$ for each case, S indicates the channel being slowly fading.

2. Calculate delay spread, rms delay spread and mean delay spread of the channel described in table 2. How wide is the channel coherence bandwidth? Is the channel frequency selective or frequency flat for GSM where the carrier spacing is 200kHz?

Solution

Mean excess delay is defined as

$$\bar{\tau} = \frac{\sum_k P(\tau_k)\tau_k}{\sum_k P(\tau_k)} \tag{29}$$

where $P(\tau_k)$ is the average absolute power of the tap corresponding to delay τ_k. For our channel we get the value $0.67\mu s$. Root mean square (rms) delay spread is defined as

$$\sigma_r^2 = \overline{(\tau^2)} + (\bar{\tau})^2 \tag{30}$$

where $\overline{(\tau^2)}$ is defined as

$$\bar{\tau} = \frac{\sum_k P(\tau_k)\tau_k}{\sum_k P(\tau_k)^2} \tag{31}$$

and the value is $1.5816\mu s$. rms delay spread for this channel is

$$\sigma_r^2 = 1.5816 + (0.67)^2 = 1.0616 \tag{32}$$

Maximum excess delay or the delay spread T_m is defined to be the delay during which multipath energy falls XdB below the maximum. In our case it would be $-10dB$. For the given channel that is the delay till the last multipath component $5\mu s$.

3. What kind of channel is frequency flat for all signals?

Solution

Table 2:

Average power [dB]	-3	0	-2	-6	-8	-10
Delay [ms]	0.0	0.2	0.5	1.6	2.3	5.0

The coherence bandwidth $(\Delta f)_c$ tells the band within the frequency components experience correlated (similar) fading. If the system bandwidth is greater than $(\Delta f)_c$ the channel is said to be frequency selective and if smaller, frequency nonselective. Frequency flat channel is fading exactly the same way for all frequency components. $(\Delta f)_c$ can be approximated as $(\Delta f)_c = 1/T_m$ thus if thye channel is just an impulse, i.e. one tap, the frequency responce is constant on all frequencies.