Digital Transmission Methods S-72.2205

Second Exercise Session

- Hypothesis Testing
- Decision Making
- Gram-Schmidt method
- Detection

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Exercise 1

We assume that under hypothesis H_1 the source output is a constant voltage m. Under hypothesis H_0 the source output is zero. Voltage is corrupted by additive noise. The output is sampled with N samples for each second. Each noise sample is modelled as zero mean Gaussian random variable with variance σ^2. The samples are assumed to be independent and identically distributed. It is known by Bayesian theory that the optimal test in such case is a likelihood ratio test compared to a decision threshold η.

a. Define the sufficient statistic of the test.

b. What is the distribution of the sufficient statistic under the two hypotheses?

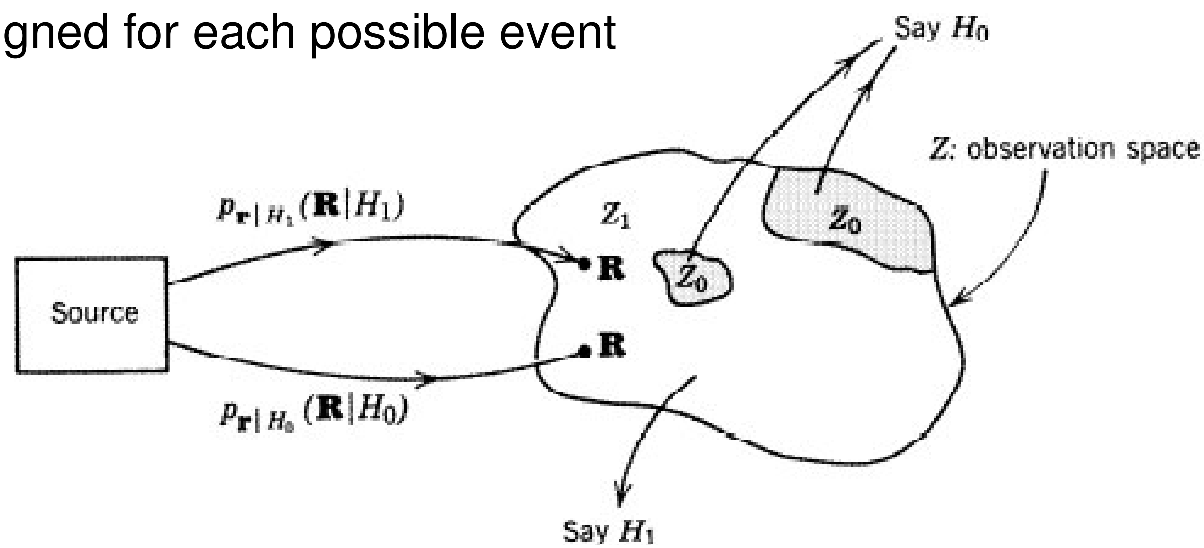
c. Calculate the detection and the false alarm probability of the optimal test.

d. What is the total error probability?

Exercise 1 Sol.

Simple binary hypothesis tests

- Only 2 possible source outputs – Hypotheses
- The transition mechanism generates points according to 2 known conditional PDFs
- Observation space is one dimensional in binary hypothesis tests
- A cost is assigned for each possible event



4 possible events
- H0 generated and H0 chosen ✓
- H0 generated and H1 chosen ✗ False alarm probability
- H1 generated and H1 chosen ✓ Detection probability
- H1 generated and H0 chosen ✗ Miss probability

Exercise 1 Sol. (contd)

The source output under the 2 hypotheses is :

$$H_1 : r_i = m + n_i, \quad i = 1, 2, \ldots, N$$
$$H_0 : r_i = n_i, \quad i = 1, 2, \ldots, N$$

The probability density of a single sample r_i under the 2 hypotheses is :

$$p_{r_i|H_0}(R_i \mid H_0) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{R_i^2}{2\sigma^2}\right)$$
$$p_{r_i|H_1}(R_i \mid H_1) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(R_i - m)^2}{2\sigma^2}\right)$$

r_i : the ith random variable

R_i: the value of the ith sample

The joint pdf of the N samples is :

$$p_{\mathbf{r}|H_0}(\mathbf{R} \mid H_0) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{R_i^2}{2\sigma^2}\right)$$
$$p_{\mathbf{r}|H_1}(\mathbf{R} \mid H_1) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(R_i - m)^2}{2\sigma^2}\right)$$

What is implied ?

$$\mathbf{R} = (R_1, R_2, \ldots, R_i, \ldots R_N)$$

Exercise 1 Sol. (contd)

Construct the likelihood ratio test of the 2 hypotheses :

$$\Lambda(\mathbf{R}) = \frac{p_{\mathbf{r}|H_1}(\mathbf{R} \mid H_1)}{p_{\mathbf{r}|H_0}(\mathbf{R} \mid H_0)} = \frac{\prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(R_i - m)^2}{2\sigma^2}\right)}{\prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{R_i^2}{2\sigma^2}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Bayesian threshold dependent on a priori prob. & decision costs

The common terms are cancelled:

$$\Lambda(\mathbf{R}) = \prod_{i=1}^{N} \exp\left(-\frac{(R_i - m)^2}{2\sigma^2} + \frac{R_i^2}{2\sigma^2}\right) = \prod_{i=1}^{N} \exp\left(\frac{2 \cdot R_i \cdot m - m^2}{2\sigma^2}\right) = \prod_{i=1}^{N} \exp\left(\frac{R_i \cdot m}{\sigma^2} - \frac{m^2}{2\sigma^2}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

The calculations are brought to the log-domain :

$$\ln \Lambda(\mathbf{R}) = \frac{m}{\sigma^2} \sum_{i=1}^{N} R_i - \frac{Nm^2}{2\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln(\eta)$$

Equivalently :

$$\frac{m}{\sigma^2} \sum_{i=1}^{N} R_i - \frac{Nm^2}{2\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln(\eta)$$

Finally :

$$\ell = \sum_{i=1}^{N} R_i \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma^2}{m} \ln(\eta) + \frac{Nm}{2} \triangleq \gamma$$

Decision threshold

Exercise 1 Sol. (contd)

a. The sufficient statistic of the test adds the N collected amplitudes and compares the sum with the decision threshold γ. If the sum is lower than the decision threshold, hypothesis H_0 is favoured. Otherwise the voltage m is assumed to be present.

b. Under hypothesis H0 the sufficient statistic is the sum of N independent Gaussian random variables with zero mean and variance σ^2. As a result the sufficient statistic follows normal distribution with zero mean and variance $N \times \sigma^2$.

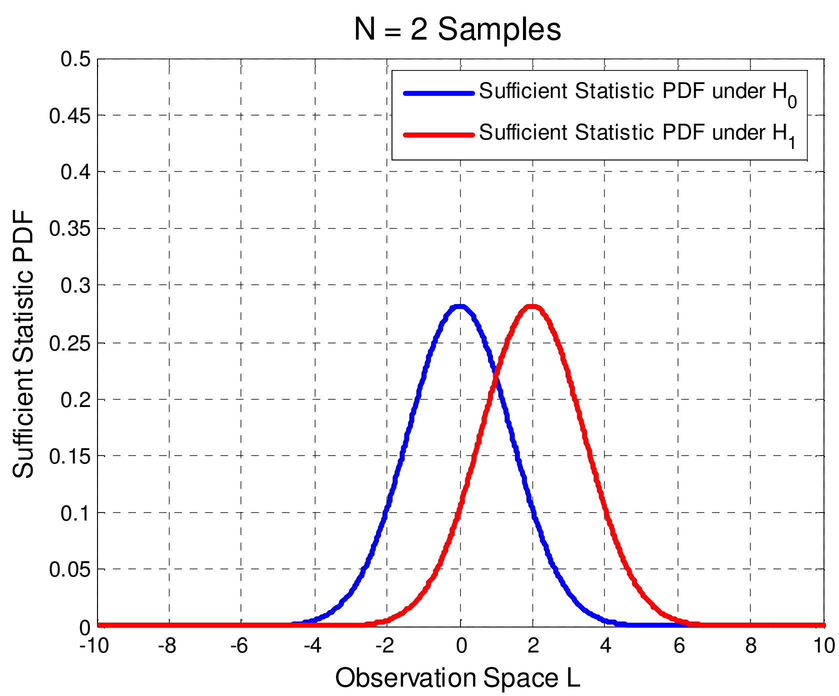
Under hypothesis H1 the sufficient statistic is the sum of N independent Gaussian random variables with mean equal to m and variance σ^2. The sufficient statistic follows normal distribution with mean equal to $N \times m$ and variance $N \times \sigma^2$.

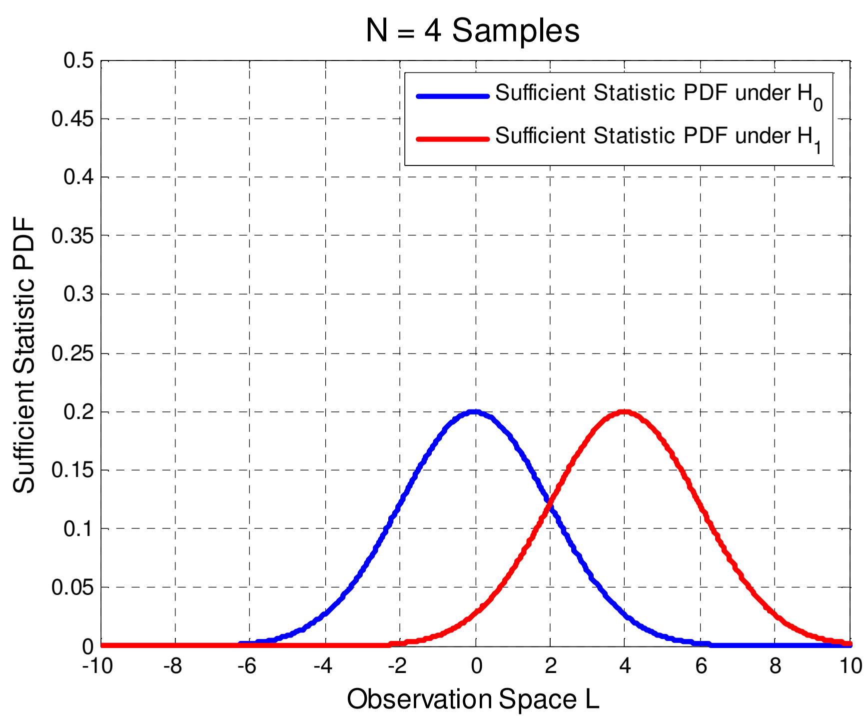
HINT!!

The sum of independent Gaussian random variables is also a Gaussian Random variable with mean the sum of individual means and variance equal to the sum of individual variances

For $m = 1$ and $\sigma^2 = 1$ we plot the PDF of the sufficient statistic under both the hypotheses and for different amount of samples

Exercise 1 Sol. (contd)





- As the number of samples increases the mean separation of the two distributions increases too
- As the number of samples increases the variance of both distributions increases.
- The probability of decision error is reduced as the number of samples increases
- How to calculate the probability of decision error?

Exercise 1 Sol. (contd)

$$Q(z) \triangleq \int_z^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$erfc(z) \triangleq \int_z^{\infty} \frac{2}{\sqrt{\pi}} \exp\left(-x^2\right) dx$$

$$Q(z) = \frac{1}{2} erfc\left(\frac{z}{\sqrt{2}}\right)$$

c1. False alarm probability : Decide that the voltage is present when it is actually absent

$$P_F = \int_{\gamma}^{\infty} p\left(L \mid H_0'\right) dL = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi N\sigma^2}} \cdot \exp\left(-\frac{L^2}{2N\sigma^2}\right) dL$$

c2. Detection probability : Decide that the voltage is present when it is actually present

$$P_D = \int_{\gamma}^{\infty} p\left(L \mid H_1'\right) dL = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi N\sigma^2}} \cdot \exp\left(-\frac{(L - N \cdot m)^2}{2N\sigma^2}\right) dL$$

d. The probability of decision error is written below :

$$\Pr(\varepsilon) \triangleq P_0 \cdot P_F + P_1 \cdot P_M$$

N = 2 Samples

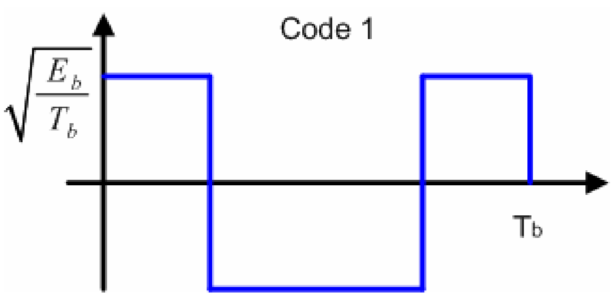
$P_F = 0.24$ $P_D = 0.76$

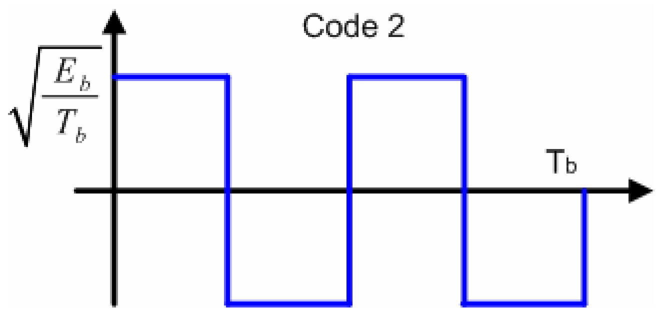
N = 4 Samples

$P_F = 0.16$ $P_D = 0.84$

Exercise 2

Consider a DS-SS system where the following Walsh-Hadamard codes of length N = 4 are used for orthogonalization.





Assume BPSK modulation scheme and symbol energy equal to Eb. The noise in the receiver is modelled as Gaussian random variable with zero mean and variance equal to σ^2

Compare the performance of the following two schemes:

Case 1 : For bit +1 the firts code is used and for bit 0 the second code is used

Case 2 : Only the first code with antipodal signalling is used both for bits 1 and 0

Exercise 2 Sol.

One can see that the two waveforms are orthogonal to each other. This means that for the performance of the two schemes we expect a result similar to the one by comparison of BFSK and BPSK.

Case 1 : Orthogonal signalling

$$H_1 : \begin{bmatrix} A & -A & -A & A \end{bmatrix}$$
$$H_2 : \begin{bmatrix} A & -A & A & -A \end{bmatrix}$$

The joint pdf of the 4 samples is :

$$H_1 : \frac{1}{\left(2\pi\sigma^2\right)^2} \cdot \exp\left(-\frac{\left(R_1 - A\right)^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{\left(R_2 + A\right)^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{\left(R_3 + A\right)^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{\left(R_4 - A\right)^2}{2\sigma^2}\right)$$

$$H_2 : \frac{1}{\left(2\pi\sigma^2\right)^2} \cdot \exp\left(-\frac{\left(R_1 - A\right)^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{\left(R_2 + A\right)^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{\left(R_3 - A\right)^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{\left(R_4 + A\right)^2}{2\sigma^2}\right)$$

The likelihood ratio of the two hypotheses is :

$$\Lambda(\mathbf{R}) = \frac{p_{\mathbf{r}|H_1}(\mathbf{R} \mid H_1)}{p_{\mathbf{r}|H_0}(\mathbf{R} \mid H_0)} = \frac{\frac{1}{\left(2\pi\sigma^2\right)^2} \cdot \exp\left(-\frac{\left(R_1 - A\right)^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{\left(R_2 + A\right)^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{\left(R_3 + A\right)^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{\left(R_4 - A\right)^2}{2\sigma^2}\right)}{\frac{1}{\left(2\pi\sigma^2\right)^2} \cdot \exp\left(-\frac{\left(R_1 - A\right)^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{\left(R_2 + A\right)^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{\left(R_3 - A\right)^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{\left(R_4 + A\right)^2}{2\sigma^2}\right)} \underset{H_2}{\overset{H_1}{\gtrless}} \eta$$

Exercise 2 Sol.

Finally :

$$\ell = \frac{1}{2}\cdot\left[R_4 - R_3\right] \underset{H_2}{\overset{H_1}{\gtrless}} \frac{\sigma^2}{4A}\ln\left(\eta\right) = \gamma$$

The distribution of the test statistic under the 2 hypotheses is :

$$H_1 : \quad \ell \sim N\left(A, \frac{\sigma^2}{2}\right)$$

$$H_2 : \ell \sim N\left(-A, \frac{\sigma^2}{2}\right)$$

For equally probable hypotheses the probability of error is :

$$\mathrm{P}\left(e\right) = \frac{1}{2}\cdot\mathrm{P}\left(e \mid H_1\right) + \frac{1}{2}\cdot\mathrm{P}\left(e \mid H_2\right) = \mathrm{P}\left(e \mid H_1\right)$$

The probability of error under hypothesis H_1 is :

$$\mathrm{P}\left(e\right) = \mathrm{P}\left(e \mid H_1\right) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2/2}}\cdot\exp\left(-\frac{\left(L-A\right)^2}{2\sigma^2/2}\right)dL = Q\left(\sqrt{\frac{2A^2}{\sigma^2}}\right)$$

Why the lower integration limit equals 0?

Under antipodal signalling we do not have the 2 terms of the likelihood ratio cancelled.

The sufficient statistic of the test contains all the 4 random variables, R_1,R_2,R_3 and R_4

After some manipulation the sufficient statistic is found to be :

Exercise 2 Sol.

$$\ell' = \frac{1}{4}\cdot\left[R_1 - R_2 - R_3 + R_4\right] \underset{H_2}{\overset{H_1}{\gtrless}} \frac{\sigma^2}{8A}\ln(\eta) = \gamma'$$

The distribution of the test statistic under the 2 hypotheses becomes :

$$H_1 : \quad \ell' \sim N\left(A, \frac{\sigma^2}{4}\right)$$

$$H_2 : \ \ell' \sim N\left(-A, \frac{\sigma^2}{4}\right)$$

By following the same procedure as previously :

$$\mathrm{P}(e) = \mathrm{P}(e \mid H_1) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2/4}}\cdot\exp\left(-\frac{(L-A)^2}{2\sigma^2/4}\right)dL = Q\left(\sqrt{\frac{4A^2}{\sigma^2}}\right)$$

Exercise 3

A digital transmission system in an AWGN channel is power limited but not bandwidth limited. The receiver power is 1pW and the one-sided noise power spectral density is 10^{-20} W/Hz.

Using orthogonal signalling where the number of symbols M is an integer power of two:

a. Determine the minimum M to transmit 10Mbps if the BER requirement equals 10^{-6}

b. Calculate how many times must the signal bandwidth be increased from the value in a. when the bit rate is doubled but the BER requirement remains unchanged.

Exercise 3 Sol.

For orthogonal signalling the upper bound for the BEP is :

$$BEP \le \frac{M}{2} Q\left(\sqrt{\frac{E}{N_o}}\right) = \frac{M}{2} Q\left(\sqrt{\frac{P_{rx}}{N_o R_s}}\right) = \frac{M}{2} Q\left(\sqrt{\frac{P_{rx}\log_2(M)}{N_o R_b}}\right)$$

For the input values :

$$\frac{P_{rx}}{N_o R_b} = \frac{10^{-12}}{10^{-20} \cdot 10^{7}} = 10 \rightarrow BEP \le \frac{M}{2} Q\left(\sqrt{10\log_2(M)}\right)$$

For different values of M :

$$M = 2 \rightarrow BEP \le \frac{M}{2} Q\left(\sqrt{10\log_2(M)}\right) = Q\left(\sqrt{10}\right) = 7.83 \cdot 10^{-4}$$

$$M = 4 \rightarrow BEP \le \frac{M}{2} Q\left(\sqrt{10\log_2(M)}\right) = 2Q\left(\sqrt{20}\right) = 7.74 \cdot 10^{-6}$$

$$M = 8 \rightarrow BEP \le \frac{M}{2} Q\left(\sqrt{10\log_2(M)}\right) = 4Q\left(\sqrt{30}\right) = 8.64 \cdot 10^{-8}$$

Exercise 3 Sol. (contd)

When the bit rate is doubled we can determine the required M-value in the same way :

$$M = 8 \rightarrow BEP \leq \frac{M}{2} \mathrm{Q}\left(\sqrt{5\log_2(M)}\right) = 4\mathrm{Q}\left(\sqrt{15}\right) = 2.15 \cdot 10^{-4}$$

$$M = 16 \rightarrow BEP \leq \frac{M}{2} Q\left(\sqrt{5\log_2(M)}\right) = 8Q\left(\sqrt{20}\right) = 3.10 \cdot 10^{-5}$$

$$M = 32 \rightarrow BEP \leq \frac{M}{2} Q\left(\sqrt{5\log_2(M)}\right) = 16Q\left(\sqrt{25}\right) = 4.59 \cdot 10^{-6}$$

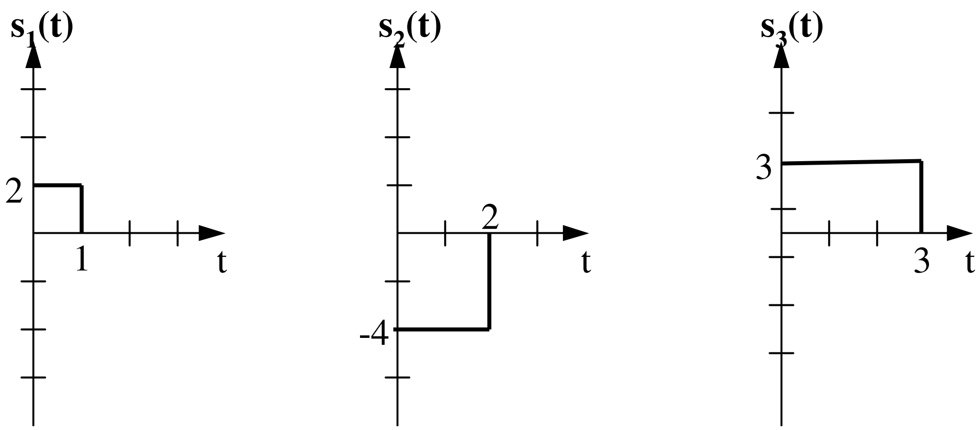
$$M = 64 \rightarrow BEP \leq \frac{M}{2} Q\left(\sqrt{5\log_2(M)}\right) = 32Q\left(\sqrt{30}\right) = 6.91 \cdot 10^{-7}$$

The required bandwidth is proportional to the number of symbol values.

Thus the increase of the bandwidth is 64/8 = 8-fold.

Exercise 4

Using the Gram-Schmidt procedure find an orthogonal basis functions set for the three signals depicted in the following figure:



Express the three signals with respect to the orthogonal basis functions set.

Exercise 4 Sol.

Gram-Schmidt orthogonalization procedure permits to:

Represent each of M energy signals as a linear combination of N orthonormal basis functions where N≤M

$$s_i(t)=\sum_{j=1}^{N} s_{ij}\phi_j(t),\quad \begin{matrix} 0\le t\le T \\ i=1,2,\ldots,M \end{matrix}$$ where $$s_{ij} \triangleq \int_0^T s_i(t)\phi_j(t)dt,\quad \begin{matrix} i=1,2,\ldots,M \\ j=1,2,\ldots,N \end{matrix}$$ and $$\int_0^T \phi_i(t)\phi_j(t)dt = \begin{cases} 1, & if\ i=j \\ 0, & if\ i\ne j \end{cases}$$

Orhtogonalization Algorithm:

–Define the first basis function as: $\phi_1(t)=\dfrac{s_1(t)}{\sqrt{E_1}}$

–Define the other basis functions as: $\phi_i(t)=\dfrac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)dt}},\ i=2,3,\ldots N$ where $g_i(t)=s_i(t)-\sum_{j=1}^{i-1} s_{ij}\phi_j(t)$

In practice:

–The energy of the first signal is: $E_1=\int_0^1 (2)^2 dt=4$

–Therefore the first basis function is: $\phi_1(t)=\dfrac{s_1(t)}{\sqrt{E_1}}=\begin{cases} 1, & if\ 0\le t\le 1 \\ 0, & otherwise \end{cases}$

Exercise 4 Sol. (contd)

Calculate:

$$s_{21} = \int_0^1 s_2(t)\phi_1(t)\,dt = \int_0^1 (-4)\cdot(1)\,dt = -4$$

Apply the formula:

$$g_2(t) = s_2(t) - s_{21}\phi_1(t) = \begin{cases} -4, & 1 \le t \le 2 \\ 0, & otherwise \end{cases} \quad \Rightarrow \quad \phi_2(t) = \begin{cases} -1, & if\ 1 \le t \le 2 \\ 0, & otherwise \end{cases}$$

Similarly:

$$s_{31} = \int_0^1 s_3(t)\phi_1(t)\,dt = \int_0^1 (3)(1)\,dt = 3 \quad \text{and} \quad s_{32} = \int_1^2 s_3(t)\phi_2(t)\,dt = \int_0^1 (3)(-1)\,dt = -3$$

$$g_3(t) = s_3(t) - s_{31}\phi_1(t) - s_{32}\phi_2(t) = \begin{cases} 3, & 2 \le t \le 3 \\ 0, & otherwise \end{cases} \quad \Rightarrow \quad \phi_3(t) = \begin{cases} 1, & if\ 2 \le t \le 3 \\ 0, & otherwise \end{cases}$$

The signals can be represented as:

$$s_1(t) = 2\phi_1(t)$$

$$s_2(t) = -4\phi_1(t) + 4\phi_2(t)$$

$$s_3(t) = 3\phi_1(t) - 3\phi_2(t) + 3\phi_3(t)$$

Remark: The sum of the squares of the coefficients equals the signal energy

Exercise 5

Consider the following signal set:

$$s_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos\left(2\pi f_c t + i\frac{\pi}{4}\right) & 0 \le t \le T \\ 0 & otherwise \end{cases} \qquad i = 1,2,3,4 \qquad f_c = {n_c}/{T}$$

a. What is the dimensionality of the signal set?

b. Find an orthogonal basis functions set for the above signal set.

c. Using the formula:

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t),\ i = 1,2,3,4$$

c1. find the coefficients s_{ij}

c2. Sketch the signal set within the orthogonal basis functions space.

Exercise 5 Sol.

- **Break the cosine into 2 sums:**

$$s_i(t)=\sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t+i\frac{\pi}{4}\right)=\sqrt{\frac{2E}{T}}\underbrace{\cos(2\pi f_c t)}\cos\left(i\frac{\pi}{4}\right)-\sqrt{\frac{2E}{T}}\underbrace{\sin(2\pi f_c t)}\sin\left(i\frac{\pi}{4}\right),\quad 0\le t\le T$$

Linear combination of 2 orthogonal function!!

- **The orthonormal functions are:**

$$\phi_1(t)=\sqrt{\frac{2}{T}}\cos(2\pi f_c t)$$

$$0\le t\le T$$

$$\phi_2(t)=\sqrt{\frac{2}{T}}\sin(2\pi f_c t)$$

- **Let's calculate s1(t):**

$$s_1(t)=\sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t+\frac{\pi}{4}\right)=\sqrt{\frac{2E}{T}}\cos(2\pi f_c t)\cos\left(\frac{\pi}{4}\right)-\sqrt{\frac{2E}{T}}\sin(2\pi f_c t)\sin\left(\frac{\pi}{4}\right)\Rightarrow$$

$$s_1(t)=\sqrt{\frac{2E}{T}}\cos(2\pi f_c t)\frac{\sqrt{2}}{2}-\sqrt{\frac{2E}{T}}\sin(2\pi f_c t)\frac{\sqrt{2}}{2}\Rightarrow$$

$$s_1(t)=\underbrace{\sqrt{E/2}}_{s11}\left[\sqrt{\frac{2}{T}}\cos(2\pi f_c t)\right]-\underbrace{\sqrt{E/2}}_{s12}\left[\sqrt{\frac{2}{T}}\sin(2\pi f_c t)\right]$$

s11 **s12**

Exercise 5 Sol. (contd)

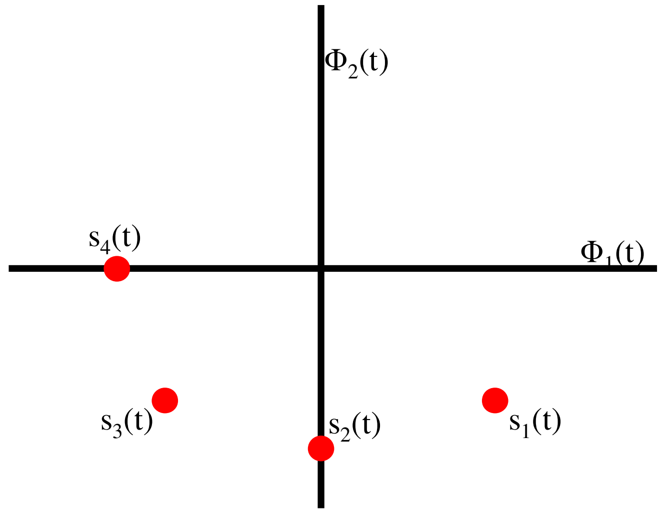
The signal coordinates in the basis functions orthogonal system are:

$$s_1(t): \quad \left(\sqrt{E/2}, -\sqrt{E/2}\right)$$

$$s_2(t): \quad \left(0, -\sqrt{E}\right)$$

$$s_3(t): \quad \left(-\sqrt{E/2}, -\sqrt{E/2}\right)$$

$$s_4(t): \quad \left(-\sqrt{E}, 0\right)$$



Exercise 6

Three messages m_1, m_2, m_3 are to be transmitted over an AWGN channel with power spectral density $N_0/2$

The messages are:

$$s_1(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & otherwise \end{cases} \qquad s_2(t) = -s_3(t) = \begin{cases} 1 & 0 \le t \le T/2 \\ -1 & T/2 \le t \le T \\ 0 & otherwise \end{cases}$$

a. What is the dimensionality of the signal space?

b. Find an appropriate basis for the signal space.

c. Draw the signal constellation for this problem

d. Derive and sketch the optimal decision regions R_1, R_2, R_3

e. Which of the three messages is more vulnerable to errors and why?
In other words which of $P(error \mid m_i \; transmitted), \; i = 1,2,3,$ is larger?

Exercise 6 Sol.

a . The dimensionality of the signal space is N=2 since $s_3(t)$ is antipodal to $s_2(t)$

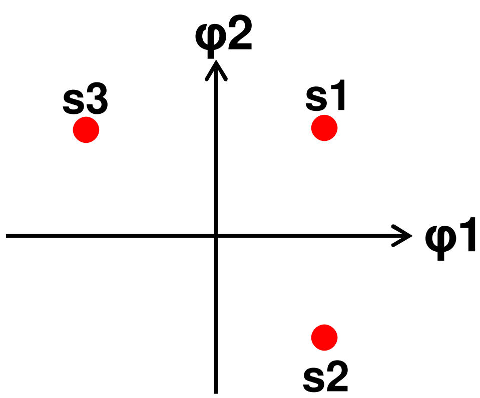
b. A simple set of othogonal functions can be 2 time-shifted rectangular pulses

c. The signal coordinates in the basis functions system are:

$$s_1 : \left(\sqrt{T/2}, \sqrt{T/2}\right)$$

$$s_2 : \left(\sqrt{T/2}, -\sqrt{T/2}\right)$$

$$s_3 : \left(-\sqrt{T/2}, \sqrt{T/2}\right)$$



Exercise 6 Sol. (contd)

d. The borders of the decision areas are calclated as:

$$P(m_1)P(\mathbf{x}|m_1) > P(m_2)P(\mathbf{x}|m_2)$$
$$P(m_1)P(\mathbf{x}|m_1) > P(m_3)P(\mathbf{x}|m_3) \Rightarrow \begin{matrix}\mathbf{x}\bullet\mathbf{m}_1 > \mathbf{x}\bullet\mathbf{m}_2 \\ \mathbf{x}\bullet\mathbf{m}_1 > \mathbf{x}\bullet\mathbf{m}_3\end{matrix}$$

$$(x_1, x_2)\bullet\left(\sqrt{T/2}, \sqrt{T/2}\right) > (x_1, x_2)\bullet\left(\sqrt{T/2}, -\sqrt{T/2}\right)$$

$$(x_1, x_2)\bullet\left(\sqrt{T/2}, \sqrt{T/2}\right) > (x_1, x_2)\bullet\left(-\sqrt{T/2}, \sqrt{T/2}\right)$$

$$x_2 > 0$$
$$x_1 > 0$$

$$P(m_2)P(\mathbf{x}|m_2) > P(m_1)P(\mathbf{x}|m_1)$$
$$P(m_2)P(\mathbf{x}|m_2) > P(m_3)P(\mathbf{x}|m_3) \Rightarrow \begin{matrix}\mathbf{x}\bullet\mathbf{m}_2 > \mathbf{x}\bullet\mathbf{m}_1 \\ \mathbf{x}\bullet\mathbf{m}_2 > \mathbf{x}\bullet\mathbf{m}_3\end{matrix}$$

$$(x_1, x_2)\bullet\left(\sqrt{T/2}, -\sqrt{T/2}\right) > (x_1, x_2)\bullet\left(\sqrt{T/2}, \sqrt{T/2}\right)$$

$$(x_1, x_2)\bullet\left(\sqrt{T/2}, -\sqrt{T/2}\right) > (x_1, x_2)\bullet\left(-\sqrt{T/2}, \sqrt{T/2}\right)$$

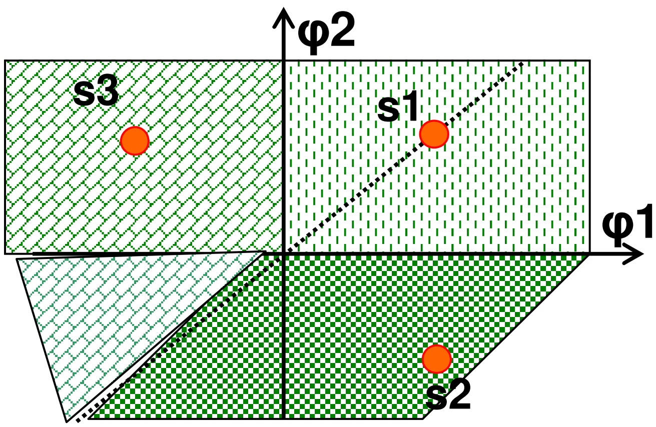
$$x_2 < 0$$
$$x_1 - x_2 > 0$$

$$P(m_3)P(\mathbf{x}|m_3) > P(m_1)P(\mathbf{x}|m_1)$$
$$P(m_3)P(\mathbf{x}|m_3) > P(m_2)P(\mathbf{x}|m_2) \Rightarrow \begin{matrix}\mathbf{x}\bullet\mathbf{m}_3 > \mathbf{x}\bullet\mathbf{m}_1 \\ \mathbf{x}\bullet\mathbf{m}_3 > \mathbf{x}\bullet\mathbf{m}_2\end{matrix}$$

$$(x_1, x_2)\bullet\left(-\sqrt{T/2}, \sqrt{T/2}\right) > (x_1, x_2)\bullet\left(\sqrt{T/2}, \sqrt{T/2}\right)$$

$$(x_1, x_2)\bullet\left(-\sqrt{T/2}, \sqrt{T/2}\right) > (x_1, x_2)\bullet\left(\sqrt{T/2}, -\sqrt{T/2}\right)$$

$$x_1 < 0$$
$$x_1 - x_2 < 0$$



Equiprobable alphabet set and same signals energy

Exercise 6 Sol. (contd)

e. Symbol error probability calculation for m1

$$P\left(e \mid m_1\right) = 1 - P\left(c \mid m_1\right) = 1 - P\left(x_1 > 0, x_2 > 0 \mid m_1\right)$$

$$P\left(e \mid m_1\right) = 1 - \left[\int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\left(x_1 - \sqrt{T/2}\right)^2}{2\sigma^2}\right) dx_1\right]^2$$

$$P\left(e \mid m_1\right) = 1 - \left[\int_{-\sqrt{T/2\sigma^2}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{t^2}{2}\right) \sigma\, dt\right]^2$$

$$P\left(e \mid m_1\right) = 1 - \left[\int_{-\sqrt{T/2\sigma^2}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]^2 \quad Note: \left(Q\left(-x\right) = 1 - Q\left(x\right)\right)$$

$$P\left(e \mid m_1\right) = 1 - \left[1 - Q\left(\sqrt{\frac{T}{N_0}}\right)\right]^2 = 2Q\left(\sqrt{\frac{T}{N_0}}\right) - Q^2\left(\sqrt{\frac{T}{N_0}}\right)$$

$$Q\left(z\right) \triangleq \int_z^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$erfc\left(z\right) \triangleq \int_z^{\infty} \frac{2}{\sqrt{\pi}} \exp\left(-x^2\right) dx$$

$$Q\left(z\right) = \frac{1}{2} erfc\left(\frac{z}{\sqrt{2}}\right)$$

Exercise 6 Sol. (contd)

e. For symbol m2 the error probability is:

$$P\left(e\,|\,m_2\right)=1-P\left(c\,|\,m_2\right)=1-P\left(x_2<0,x_1-x_2>0\,|\,m_2\right)$$

$$P\left(e\,|\,m_2\right)=1-\int_{-\infty}^{0}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{\left(x_2-\sqrt{T/2}\right)^2}{2\sigma^2}\right)dx_2\bullet\int_{0}^{\infty}\frac{1}{\sqrt{4\pi\sigma^2}}\exp\left(-\frac{\left(y-\sqrt{2T}\right)^2}{4\sigma^2}\right)dy$$

$$P\left(e\,|\,m_2\right)=1-\int_{-\infty}^{-\sqrt{T/2\sigma^2}}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{t^2}{2}\right)\sigma\,dt\bullet\int_{-\sqrt{T/\sigma^2}}^{\infty}\frac{1}{\sqrt{4\pi\sigma^2}}\exp\left(-\frac{u^2}{2}\right)\sqrt{2}\sigma\,du$$

$$y=x_1-x_2\sim N\left(\sqrt{2T},2\sigma^2\right)$$

$$P\left(e\,|\,m_2\right)=1-Q\left(\sqrt{\frac{T}{N_0}}\right)\cdot\left[1-Q\left(\sqrt{\frac{2T}{N_0}}\right)\right]=1-Q\left(\sqrt{\frac{T}{N_0}}\right)+Q\left(\sqrt{\frac{T}{N_0}}\right)Q\left(\sqrt{\frac{2T}{N_0}}\right)$$

Exercise 7

Consider a set of M signal waveforms $s_m(t), 1 \le m \le M, 0 \le t \le T$ all of which have the same energy E. The signal waveforms are separated in frequency:

$$s_m(t) = \sqrt{\frac{2E_s}{T}} \cos\left(2\pi f_c t + 2\pi \Delta f\, t\right), \quad m = 0,1,\ldots,M-1, \quad 0 \le t \le T$$

a. For which frequencies df the above signal set is orthogonal?

b. Assume that Sm(t) are orthogonal and define a new set of M waveforms as:

$$s'_m(t) = s_m(t) - \frac{1}{M}\sum_{k=1}^{M} s_k(t), \ 1 \le m \le M, \ 0 \le t \le T$$

Show that the M signal waveforms have equal energy given by: $E' = \frac{(M-1)E}{M}$

c. Show that the signals are now correlated with correlation coefficient:

$$\rho_{mn} = \frac{1}{E'}\int_0^T s'_m(t)\, s'_n(t)\, dt = -\frac{1}{M-1}$$

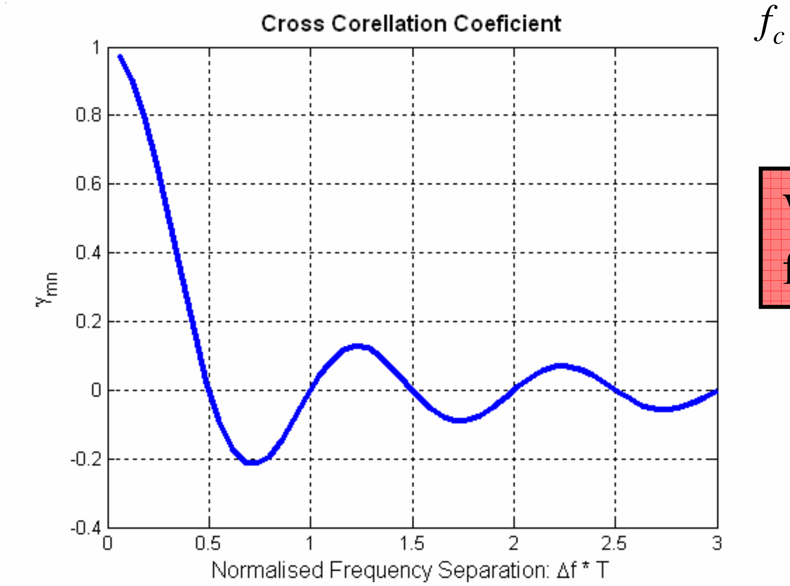
Exercise 7 Sol.

The frequency separation df determines the degree to which we can disciminate among the M possible transmitted signals.

$$\gamma_{mn} \triangleq \frac{1}{E_s}\int_0^T s_m(t)\,s_n(t)\,dt = \frac{1}{E_s}\int_0^T \frac{2E_s}{T}\cos\left(2\pi f_c t + 2\pi m\Delta f\, t\right)\cos\left(2\pi f_c t + 2\pi n\Delta f\, t\right)dt \Rightarrow$$

$$\gamma_{mn} = \frac{1}{T}\int_0^T \cos\left(2\pi(m-n)\Delta f\, t\right)dt + \frac{1}{T}\int_0^T \underline{\cos\left(4\pi f_c t + 2\pi(m+n)\Delta f\, t\right)}\,dt = \frac{\sin\left(2\pi(m-n)\Delta f\, T\right)}{2\pi(m-n)\Delta f\, T}$$

$$f_c \gg 1/T$$



We look for neighboring frequency separation

m-n=1

The signal waveforms are orthogonal when the separation between neighboring frequencies is a multiple of 1/2T

The minimum frequency separation for rthogonality between neighboring frequencies is: df = 1/2T

Exercise 7 Sol. (contd)

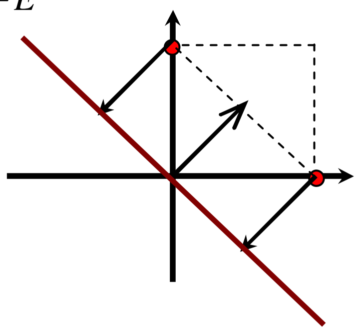
The energy of the signal waveform $s'_m(t)$ is

$$E' = \int_0^T \left| s'_m(t) \right|^2 dt = \int_0^T \left| s_m(t) - \frac{1}{M}\sum_{k=1}^{M} s_k(t) \right|^2 dt = \int_0^T s_m^2(t)dt + \frac{1}{M^2}\int_0^T \sum_{k=1}^{M}\sum_{n=1}^{M} s_k(t)s_n(t)dt - \frac{2}{M}\int_0^T s_m(t)\sum_{k=1}^{M} s_k(t)dt \Rightarrow$$

The multiple is 0 if k≠n

$$E' = E + \frac{1}{M^2}ME - \frac{2}{M}E = E - \frac{E}{M} = \frac{(M-1)}{M}E$$

- ✓ Less energy
- ✓ Less dimensions



The correlation coefficient is given by:

$$\rho_{mn} = \frac{1}{E'}\int_0^T s'_m(t)s'_n(t)dt = \frac{1}{E'}\int_0^T \left(s_m(t) - \frac{1}{M}\sum_{k=1}^{M} s_k(t) \right) \cdot \left(s_n(t) - \frac{1}{M}\sum_{l=1}^{M} s_l(t) \right) dt \Rightarrow$$

$$\rho_{mn} = \frac{1}{E'}\left[\int_0^T s_m(t)s_n(t)dt - \frac{1}{M}\int_0^T s_m(t)\sum_{l=1}^{M} s_l(t)dt - \frac{1}{M}\int_0^T s_n(t)\sum_{k=1}^{M} s_k(t)dt + \frac{1}{M^2}\int_0^T \sum_{k=1}^{M} s_k(t)\sum_{l=1}^{M} s_l(t)dt \right] \Rightarrow$$

$$\rho_{mn} = \frac{1}{E'}\left[0 - \frac{1}{M}E - \frac{1}{M}E + \frac{1}{M^2}ME \right] = -\frac{1}{M}\cdot\frac{E}{E'} = -\frac{1}{M}\cdot\frac{M}{M-1} = -\frac{1}{M-1}$$

Evaluate both results for M=2

Appendix

In binary Bayesian hypotheses testing problems it is decided that hypothesis is present in a way to minimize the Bayesian risk:

$$R = C_{00} p_0 \Pr\left(say\ H_0 \mid H_0\ is\ true\right) + C_{10} p_0 \Pr\left(say\ H_1 \mid H_0\ is\ true\right)$$
$$+ C_{11} p_1 \Pr\left(say\ H_1 \mid H_1\ is\ true\right) + C_{01} p_1 \Pr\left(say\ H_0 \mid H_1\ is\ true\right)$$

Given the observation vector , show that the division of the observation space minimizing the Bayesian risk is the following test:

$$say\ H_0 \quad if\ \Lambda\left(\mathbf{r}\right) < \eta \qquad\qquad \Lambda\left(\mathbf{r}\right) = \frac{p_{\mathbf{R}|H_1}\left(\mathbf{r} \mid H_1\right)}{p_{\mathbf{R}|H_0}\left(\mathbf{r} \mid H_0\right)} \qquad\qquad \eta = \frac{p_0\left(C_{10} - C_{00}\right)}{p_1\left(C_{01} - C_{11}\right)}$$
$$say\ H_1 \quad if\ \Lambda\left(\mathbf{r}\right) > \eta$$

Assume that under hypothesis H0 the generated signal is absent and the received signal is:

$$x\left(t\right) = w\left(t\right)$$

where w(t) stands for one dimensional Gaussian noise with zero mean and variance.
Under hypothesis H1 the generated signal is present and the received signal is:

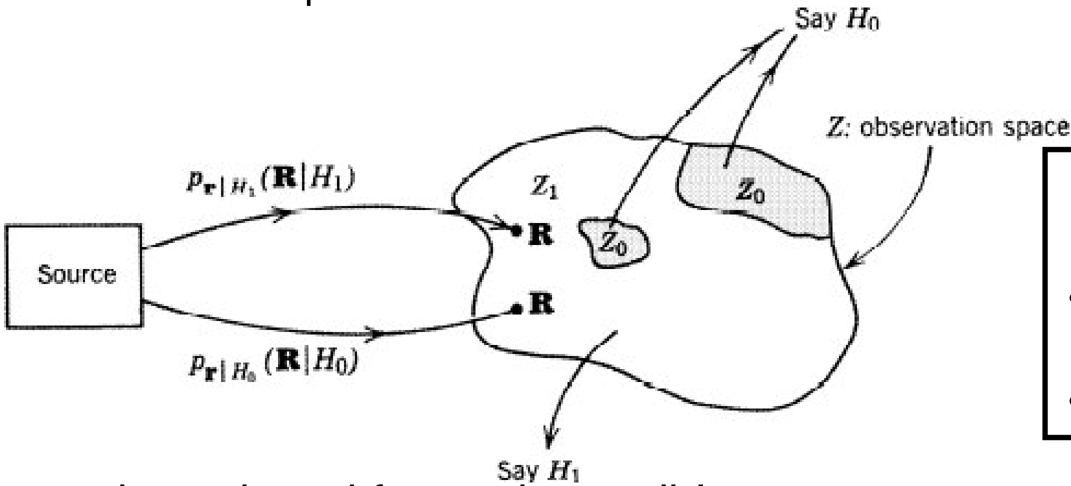
$$x\left(t\right) = w\left(t\right) + s\left(t\right)$$

where s(t) stands for constant DC level A. For equal probable hypothesis assume unity amplitude and noise variance and evaluate the error probability with respect to the value of the decision threshold. For which value of the decision threshold the error probability becomes minimal?

Appendix

Simple binary hypothesis tests

- Only 2 possible source outputs according to a known PDF
  - In general the generated hypothesis is N-dimensional
- The transition mechanism generates points according to 2 known conditional PDFs
  - The received points have N coordinates



4 possible events

- H0 generated and H0 chosen ✓
- H0 generated and H1 chosen ×
- H1 generated and H1 chosen ✓
- H1 generated and H0 chosen ×

→ A cost is assigned for each possible event

→ The mean value of the cost on average must be as small as possible (Bayes Criterion)

Problem Formulation:

How to use the a priori knowledge for hypotheses generation and probabilistic transition mechanism in order to divide optimally the observation space.

Appendix

Write an expression for the mean value of the cost including the decision regions and the transition probabilities:

$$\mathcal{R} = C_{00}P_0 \int_{Z_0} p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\, d\mathbf{R} + C_{10}P_0 \int_{Z_1} p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\, d\mathbf{R} + C_{11}P_1 \int_{Z_1} p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)\, d\mathbf{R} + C_{01}P_1 \int_{Z_0} p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)\, d\mathbf{R}.$$

$$\mathcal{R} = P_0C_{00} \int_{Z_0} p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\, d\mathbf{R} + P_0C_{10} \int_{Z-Z_0} p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\, d\mathbf{R} + P_1C_{01} \int_{Z_0} p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)\, d\mathbf{R} + P_1C_{11} \int_{Z-Z_0} p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)\, d\mathbf{R}.$$

Use that:

$$\int_Z p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\, d\mathbf{R} = \int_Z p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)\, d\mathbf{R} = 1,$$

Note!! C01>C11 & C10>C00
The 2 terms inside the integral are positive

End up with:

$$\mathcal{R} = P_0C_{10} + P_1C_{11} + \int_{Z_0} \{[P_1(C_{01} - C_{11})p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)] - [P_0(C_{10} - C_{00})p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)]\}\, d\mathbf{R}.$$

fixed cost

Choose Z0 in a way to minimize the integral

Appendix

To minimize the integral the observation space points that make it positive must be assigned to Z1. Say H1 when:

$$P_1\left(C_{01}-C_{11}\right)p_{\mathbf{r}|H_1}\left(\mathbf{R}\,|\,H_1\right)\geq P_0\left(C_{10}-C_{00}\right)p_{\mathbf{r}|H_0}\left(\mathbf{R}\,|\,H_0\right)$$

Equivalently:

$$\frac{p_{\mathbf{r}|H_1}\left(\mathbf{R}\,|\,H_1\right)}{p_{\mathbf{r}|H_0}\left(\mathbf{R}\,|\,H_0\right)}\overset{H_1}{\geq}\frac{P_0\left(C_{10}-C_{00}\right)}{P_1\left(C_{01}-C_{11}\right)}$$

For one dimensional binary hypothesis testing the observation space is linear

$$P_e=P_0\Pr\left(say\ H_1\,|\,H_0\ is\ true\right)+P_1\Pr\left(say\ H_0\,|\,H_1\ is\ true\right)$$

$$P_e=P_0\Pr\left(false\ alarm\right)+P_1\Pr\left(miss\right)$$

$$P_e=P_0\Pr\left(false\ alarm\right)+P_1\left(1-\Pr\left(\det ection\right)\right)$$

$$P_e=\frac{1}{2}\cdot\frac{1}{2}erfc\left(\frac{\gamma}{\sqrt{2}\sigma}\right)+\frac{1}{2}\cdot\frac{1}{2}\left(1-erfc\left(\frac{\gamma-A}{\sqrt{2}\sigma}\right)\right)$$