Lecture 2
M/G/1 queues

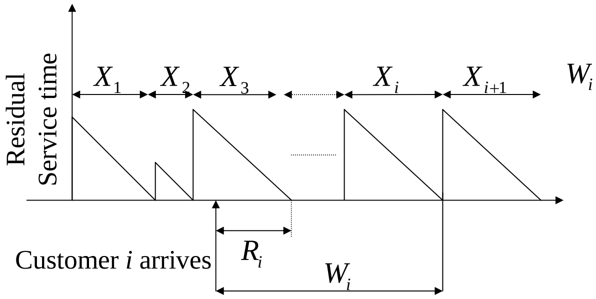
M/G/1-queue

- Poisson arrival process
- Arbitrary service time distribution
- Single server
- To determine the state of the system at time *t*, we must know
  - The number of customers in the systems $N(t)$
  - Time that the customer currently being served has already been served (Markov property does not apply) $X_i(t) - R_i(t)$
  - The M/G/1 queue can be modeled with Imbedded Markov chain by sampling at time instances when the customers depart from the system

M/G/1-queue

- Define

W_i	Waiting (queuing) time of customer *i*
R_i	Residual service time when customer *i* arrives
X_i	Service time of customer *i*
N_i	Number of customers in the system upon arrival of customer *i*.



$$W_i = R_i + \sum_{j=i-N_i}^{i-1} X_i$$

M/G/1-queue

- Let us consider mean over the customers
  $i = 1, 2, \ldots, \infty$

$$E\{W_i\} = E\{R_i\} + E\left\{\sum_{j=i-N_i}^{i-1} X_i\right\}$$

$$W = \lim_{i\to\infty} E\{W_i\} = \lim_{i\to\infty} E\{R_i\} + \lim_{i\to\infty} E\left\{\sum_{j=i-N_i}^{i-1} X_i\right\}$$

- Assume that the service times are mutually independent with the first two moments given by

$$\overline{X} = E\{X_i\} = \frac{1}{\mu}, \quad \overline{X^2} = E\{X_i^2\}$$

M/G/1-queue

- Aggregate service time

$$E\left\{\sum_{j=i-N_i}^{i-1} X_i\right\} = E_{N_i}\left\{\sum_{j=i-N_i}^{i-1} E\left\{X_i \middle| N_i\right\}\right\} = N_i\,\overline{X}$$

$\lim_{i\to\infty} N_i = N_Q$ Expected number of users in the queue

  - Litle's theorem $N_Q = \lambda W$

M/G/1-queue

- Residual service time at time t

$$r(t) = X_i - \left(t - \tau_i\right), \quad \tau_i \le t \le \tau_i + X_i$$

- Mean residual time

$0 \qquad t \to \infty$

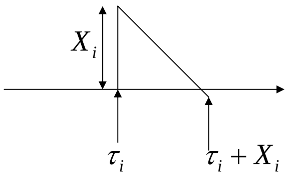
$$\left\langle r(t) \right\rangle = \frac{1}{t}\int_0^t r(t)dt = \frac{1}{t}\sum_{i=1}^{M(t)} \frac{1}{2} X_i^2 + X_{M(t)+1} - \left(t - \tau_{M(t)+1}\right)$$

$M(t)$ denotes the number of customers served during the time interval $(0,t)$

$$\overline{R} = \lim_{t\to\infty} \left\langle r(t) \right\rangle = \lim_{t\to\infty} \frac{1}{t}\int_0^t r(t)dt$$

$$= \lim_{t\to\infty} \frac{1}{t}\sum_{i=1}^{M(t)} \frac{1}{2} X_i^2 = \lim_{t\to\infty} \frac{1}{2} \underbrace{\frac{M(t)}{t}}_{\to\lambda} \underbrace{\frac{\sum_{i=1}^{M(t)} X_i^2}{M(t)}}_{\to\overline{X^2}}$$

$$= \frac{1}{2}\lambda\,\overline{X^2}$$



In steady state the number of arriving customers must be equal to the number of departing customers

M/G/1-queue

- Expected waiting (queuing) time

$$E\{W_i\} = E\{R_i\} + E\left\{\sum_{j=i-N_i}^{i-1} X_i\right\}$$

$$W = \lim_{i\to\infty} E\{W_i\} = \frac{1}{2}\lambda\overline{X^2} + \lambda\overline{X}\,W$$

$$\boxed{W = \frac{1}{2}\frac{\lambda\overline{X^2}}{1-\rho}, \quad \rho = \lambda\overline{X} \quad \text{Pollaczek-Khinchin}}$$

- Packet delay = waiting time + service time

$$T = W + \overline{X} = \overline{X} + \frac{1}{2}\frac{\lambda\overline{X^2}}{1-\rho}, \quad \rho = \lambda\overline{X}$$

Imbedded Markov Chain

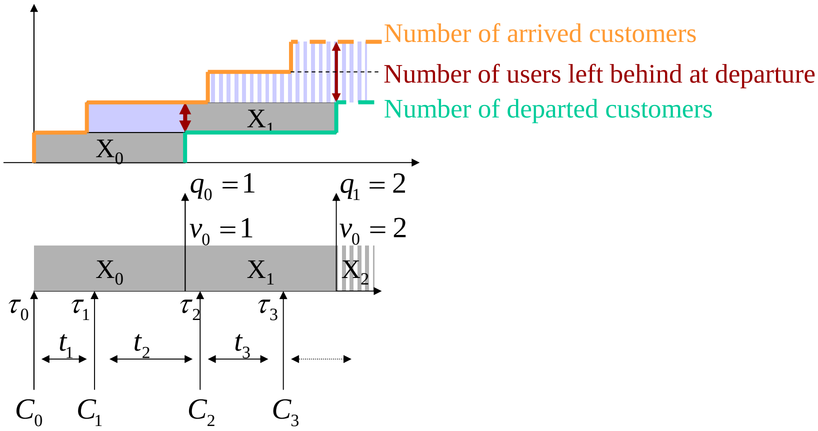
- Define

C_i	Customer *i*
τ_i	Arrival time of customer *i*
$t_i = \tau_i - \tau_{i-1}$	Interarrival time between customers *i*-1 and *i*.
X_i	Service time of customer *i*
q_i	Number of users left behind by departure of customer *i*
$v_i = N(X_i)$	Number of customers arriving during the service of customer *i*

- **Imbedded Markov chain:** The system is sampled on those time instants when customers depart from it. The state of the system on those time instances depend only on the number of customers in the system.

Imbedded Markov Chain

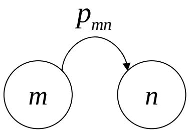
- M/G/1



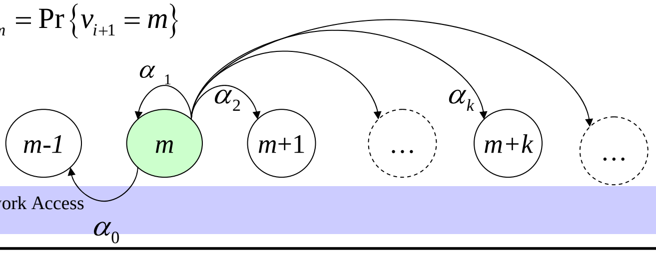
Transition probabilities

- Transition probability

$$p_{mn} = \Pr\{q_{i+1} = n | q_i = m\}$$



- Since transitions are observed only at departures, it is clear that $q_{i+1} \ge q_i - 1$
- Transitions are possible from state k to any state k-1, k k+1, k+2, k+3,…
- The transition is determined by the number of arrivals during the service of the customer i: v_i.
- Let $\alpha_m = \Pr\{v_{i+1} = m\}$



Transition probabilities

- Let b(x) denote the probability density function of the service time X (assumed to be equal for all i)
- The arrival process is Poisson

$$\Pr\{v_{i+1}=k \mid X_{i+1}\} = \frac{(\lambda X_{i+1})^k}{k!} e^{-\lambda X_{i+1}}$$

- Probability that there is k arrivals during the service time of customer $i+1$ is then

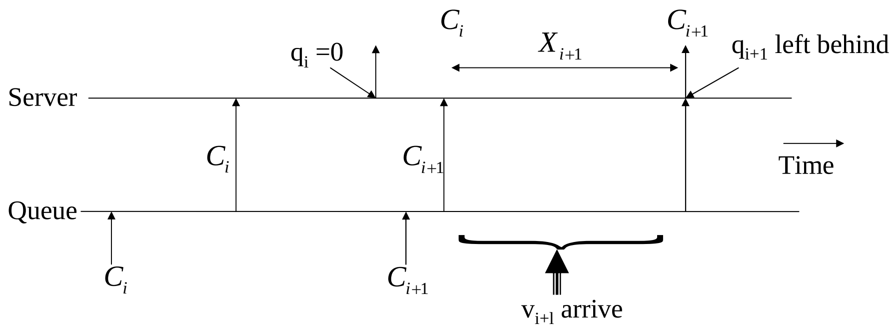
$$\alpha_k = \Pr\{v_{i+1}=k\} = \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} b(x)dx$$

- Probability transitions are independent of i

$$p_{mn} = \begin{cases} 0 & n < m-1 \\ \alpha_0 & n = m-1 \\ \alpha_1 & n = m \\ \alpha_{m-n+1} & n > m \end{cases}$$

Mean queue length

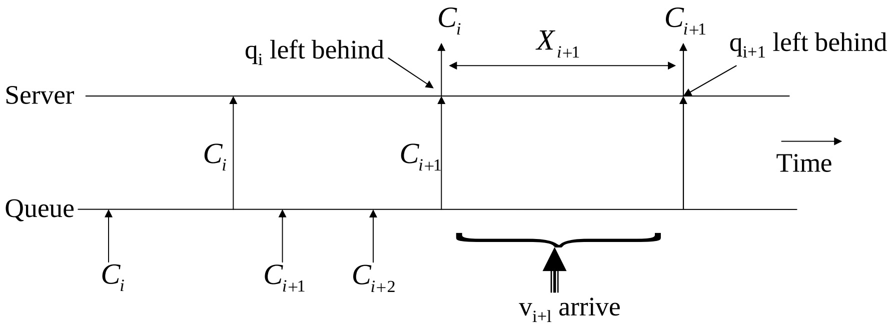
- Case 1: $q_i=0$ Departure of customer i leaves non-empty system



$$q_{i+1} = v_{i+1}$$

Mean queue length

- Case 2: $q_i>0$ Departure of customer i leaves non-empty system



$$q_{i+1} = q_i - 1 + v_{i+1}$$

Mean queue length

- Hence, the number of customers left behind by customer i+1 is

$$q_{i+1} = \begin{cases} q_i - 1 + v_{i+1} & q_i > 0 \\ v_{i+1} & q_i = 0 \end{cases}$$

- This can be written as

$$q_{i+1} = q_i - \Delta_{q_i} + v_{i+1}$$

where

$$\Delta_i = \begin{cases} 1 & i = 1, 2, 3, \ldots \\ 0 & otherwise \end{cases}$$ Discrete step function

Mean queue length

- Generating function of the arrival process during the service of customer i+1

$$V_{i+1}(z)=E\left\{z^{v_{i+1}}\right\}$$

- Characteristic function of the number of customers left behind by customer i+1

$$Q_{i+1}(z)=E\left\{z^{q_{i+1}}\right\}=E\left\{z^{q_i-\Delta_{q_i}+v_{i+1}}\right\}=E\left\{z^{q_i-\Delta_{q_i}}\right\}E\left\{z^{v_{i+1}}\right\}$$

$$=V_{i+1}(z)E\left\{z^{q_i-\Delta_{q_i}}\right\}$$

Mean queue length

- Let us first solve

$$E\left\{z^{q_i-\Delta_{q_i}}\right\}=\sum_{k=0}^{\infty}\Pr\left\{q_i=k\right\}z^{k-\Delta_k}=\Pr\left\{q_i=0\right\}z^0+\sum_{k=1}^{\infty}\Pr\left\{q_i=k\right\}z^{k-1}$$

$$=\Pr\left\{q_i=0\right\}z^0+z^{-1}\left(\sum_{k=0}^{\infty}\Pr\left\{q_i=k\right\}z^k-\Pr\left\{q_i=0\right\}z^0\right)$$

$$=\Pr\left\{q_i=0\right\}+z^{-1}\left(Q_i(z)-\Pr\left\{q_i=0\right\}\right)$$

- Hence

$$Q_{i+1}(z)=V_{i+1}(z)\left(\Pr\left\{q_i=0\right\}+z^{-1}\left(Q_i(z)-\Pr\left\{q_i=0\right\}\right)\right)$$

Mean queue length

- Steady state values

$$\tilde{q} = \lim_{i\to\infty} q_i$$
$$\tilde{v} = \lim_{i\to\infty} v_i$$
$$E\left\{\tilde{v}\right\} = \lambda\bar{X} = \rho$$

- Let us take the expected value of $q_{i+1} = q_i - \Delta_{q_i} + v_{i+1}$

$$E\left\{q_{i+1}\right\} = E\left\{q_i\right\} - E\left\{\Delta_{q_i}\right\} + E\left\{v_{i+1}\right\}$$
$$\to E\left\{\tilde{q}\right\} = E\left\{\tilde{q}\right\} - E\left\{\Delta_{\tilde{q}}\right\} + E\left\{\tilde{v}\right\}$$
$$\Rightarrow E\left\{\Delta_{\tilde{q}}\right\} = E\left\{\tilde{v}\right\}$$
$$E\left\{\Delta_{\tilde{q}}\right\} = \sum_{k=0}^{\infty} \Delta_k \Pr\left\{\tilde{q} = k\right\} = \sum_{k=1}^{\infty} \Pr\left\{\tilde{q} = k\right\} = 1 - \Pr\left\{\tilde{q} = 0\right\}$$
$$\Pr\left\{\tilde{q} = 0\right\} = 1 - E\left\{\tilde{v}\right\} = 1 - \rho$$

Mean queue length

- Steady state characteristic function

$$Q(z) = \lim_{i\to\infty} Q_i(z)$$
$$V(z) = \lim_{i\to\infty} V_i(z)$$

- Recall that

$$Q_{i+1}(z) = V_{i+1}(z)\left(\Pr\left\{q_i = 0\right\} + z^{-1}\left(Q_i(z) - \Pr\left\{q_i = 0\right\}\right)\right)$$

- Hence,

$$Q(z) = V(z)\left(\Pr\left\{\tilde{q} = 0\right\} + z^{-1}\left(Q(z) - \Pr\left\{\tilde{q} = 0\right\}\right)\right)$$
$$= V(z)\frac{(1-\rho)(1-z^{-1})}{1 - z^{-1}V(z)}$$

Mean queue length

- Characteristic function of the arrival process

$$V(z)=\sum_{k=0}^{\infty}\Pr\{\tilde{v}=k\}\,z^k=\sum_{k=0}^{\infty}\left(\int_0^{\infty}\frac{(\lambda x)^k}{k!}e^{-\lambda x}b(x)dx\right)z^k$$

$$=\int_0^{\infty}\sum_{k=0}^{\infty}\frac{(\lambda xz)^k}{k!}e^{-\lambda x}b(x)dx=\int_0^{\infty}e^{-(\lambda-\lambda z)x}b(x)dx$$

- Laplace transform of the service time distribution

$$B^*(s)=\int_0^{\infty}e^{-sx}b(x)dx \qquad s=\lambda-\lambda z$$

- Hence

$$V(z)=B^*(\lambda-\lambda z)$$

Mean queue length

- Finally, we can express the characteristic function of queue size in terms of the characteristic function of the service time:

$$Q(z)=V(z)\frac{(1-\rho)(1-z)}{V(z)-z}$$

$$V(z)=B^*(\lambda-\lambda z)$$

$$\boxed{\Rightarrow Q(z)=B^*(\lambda-\lambda z)\frac{(1-\rho)(1-z)}{B^*(\lambda-\lambda z)-z}}$$

Pollaczek-Khinchin transform equation

- Steady state probabilities can be obtained using inverse transform

$$p_k=\Pr\{\tilde{q}=k\}=Z^{-1}\{Q(z)\}$$

Example: M/M/1

- Service time distribution is exponential

$$b(x) = \mu e^{-\mu x}$$

- Laplace transform of the service time pdf

$$B^*\left(s\right) = \int_0^\infty e^{-sx} \mu e^{-\mu x} dx = \mu \int_0^\infty e^{-(\mu+s)x} dx$$

$$= \frac{\mu}{s+\mu}$$

- and arrivals

$$V(z) = B^*\left(\lambda - \lambda z\right) = \frac{\mu}{\lambda - \lambda z + \mu}$$

Example: M/M/1

- Characteristic function of queue size

$$Q(z) = V(z)\frac{\left(1-\rho\right)(1-z)}{V(z)-z} = B^*\left(\lambda-\lambda z\right)\frac{\left(1-\rho\right)(1-z)}{B^*\left(\lambda-\lambda z\right)-z}$$

$$Q(z) = \frac{\mu}{\lambda-\lambda z+\mu}\frac{\left(1-\rho\right)(1-z)}{\dfrac{\mu}{\lambda-\lambda z+\mu}-z} = \frac{1-\rho}{1-\rho z}$$

- Inverse transform yields

$$p_k = \Pr\left\{\tilde{q} = k\right\} = Z^{-1}\left\{Q(z)\right\} = \left(1-\rho\right)\rho^k$$

Distribution of waiting time

- The total time spent in the system is

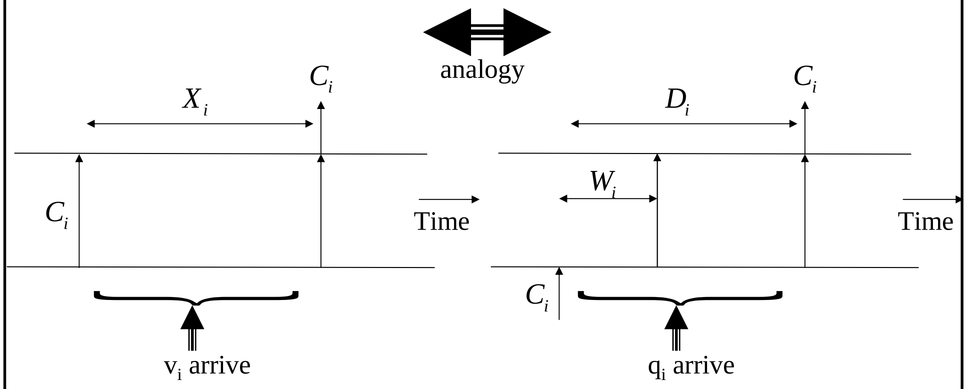
  $D_i = W_i + X_i$
- Consider FIFO queuing policy. In that case, all the customers that were in the system when customer i arrived must be served before i is served. Hence, the total number customer i leaves behind must be the number of customers that arrived during his stay in the system.

Distribution of waiting time

- Number of arrivals during X

  $V(z) = B^*\left(\lambda - \lambda z\right)$
- Number of arrivals during S

  $Q(z) = D^*\left(\lambda - \lambda z\right)$



Distribution of waiting time

- Distribution of waiting time

$$Q(z) = D^*(\lambda - \lambda z)$$

$$D^*(\lambda - \lambda z) = Q(z) = B^*(\lambda - \lambda z)\frac{(1-\rho)(1-z)}{B^*(\lambda - \lambda z) - z}$$

Let

$$s = \lambda - \lambda z$$

$$z = 1 - \frac{s}{\lambda}$$

$$\boxed{D^*(s) = B^*(s)\frac{s(1-\rho)}{s - \lambda + \lambda B^*(s)}}$$

Distribution of waiting time

- Taking the limit

$$S_i = W_i + X_i \rightarrow \tilde{S} = \tilde{W} + \tilde{X}$$

- Probability distribution of sum of two random variables is a convolution of the two probability distribution functions.
- Fourier transform of convolution integral is product of two transforms. Hence,

$$D^*(s) = W^*(s)B^*(s)$$

- It follows that

$$W^*(s) = \frac{D^*(s)}{B^*(s)} = \frac{s(1-\rho)}{s - \lambda + \lambda B^*(s)}$$

$$W^*(s) = \frac{(1-\rho)}{1 - \rho\left[\frac{1 - B^*(s)}{s\bar{X}}\right]}$$

Characteristic function of **residual service time** distribution

Residual life

- Residual life

$$R^*(s)=\frac{1-B^*(s)}{s\overline{X}}$$

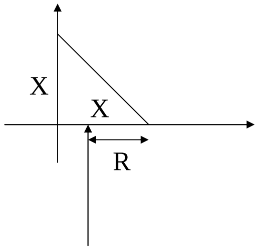
Laplace transform of the residual life distribution

$$F_R(r)=\frac{1-F_X(r)}{\overline{X}}$$

Cdf of the residual life distribution

$$E\{R\}=\frac{1}{2}\frac{\overline{X^2}}{\overline{X}}$$

$$E\{R^2\}=\frac{1}{3}\frac{\overline{X^3}}{\overline{X}}$$



Example:M/M/1

- Characteristic function of number of customers

$$B^*(s)=\int_0^\infty e^{-sx}\mu e^{-\mu x}dx=\mu\int_0^\infty e^{-(\mu+s)x}dx$$

$$=\frac{\mu}{s+\mu}$$

$$W^*(s)=\frac{s(1-\rho)}{s-\lambda+\lambda B^*(\omega)}=(1-\rho)+\frac{\lambda(1-\rho)}{s+\mu(1-\rho)}$$

- Probability distribution function

$$f_W(x)=\int_0^\infty W^*(s)e^{is}ds$$

Inverse Laplace transform

$$f_W(x)=(1-\rho)\left(\delta(x)+\lambda e^{-\mu(1-\rho)x}\right),\quad x\ge 0$$

Example:M/M/1

- Cumulative probability distribution function

$$F_W(y) = \Pr\{W \le y\} = \int_0^y (1-\rho)\left(\delta(x) + \lambda e^{-\mu(1-\rho)x}\right)dy$$

$$= 1 - \rho e^{-\mu(1-\rho)y}, \quad y \ge 0$$