PRELIMINARY PROBLEMS

P1 (Least squares channel estimation)

Suppose we model the unknown communication channel as a discrete-time FIR filter with tap coefficients $\mathbf{h} = \begin{bmatrix} h_0 & h_1 & \cdots & h_L \end{bmatrix}^T$. Let $a_0, a_1, \ldots$ denote the transmitted symbols[1] at times $n = 0,1,\ldots$. The output of the channel $y_L, y_{L+1}, \ldots$ is a convolution sum (discard the first L transient samples):

$$\begin{aligned} y_L &= a_L h_0 + a_{L-1} h_1 + \ldots + a_0 h_L + n_0 \\ y_{L+1} &= a_{L+1} h_0 + a_L h_1 + \ldots + a_1 h_L + n_1 \\ &\vdots \\ y_{L+P-1} &= a_{L+P-1} h_0 + a_{L+P-2} h_1 + \ldots + a_{P-1} h_L + n_{P-1} \end{aligned}$$

or in matrix form

$$\mathbf{y} = \mathbf{A}\mathbf{h} + \mathbf{n}, \qquad \text{(P1)}$$

where $\mathbf{n}$ is a noise vector.

In this preliminary problem we first derive the least squares solution $\hat{\mathbf{h}}_{ls}$ of (P1) and then examine its properties.

a) Show that the $\mathbf{h}$ that minimizes the squared error $J(\mathbf{h}) = \|\mathbf{y} - \mathbf{A}\mathbf{h}\|^2$ is

$$\hat{\mathbf{h}}_{ls} = \left(\mathbf{A}^T \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{y}. \qquad \text{(P2)}$$

Recall that $\nabla_{\mathbf{x}} \left\{\mathbf{c}^T \mathbf{x}\right\} = \mathbf{c}$ and $\nabla_{\mathbf{x}} \left\{\mathbf{x}^T \mathbf{B} \mathbf{x}\right\} = 2\mathbf{B}\mathbf{x}$. $\mathbf{c}$ and $\mathbf{x}$ are column vectors, $\mathbf{B}$ is a symmetric matrix.

b) (recap on linear algebra) Consider the system of linear equations:

[1] These symbols are assumed known.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

or in matrix form $\mathbf{Ax} = \mathbf{b}$. Under what conditions on b_1, b_2, b_3 is the system solvable? Find $\mathbf{x}$ in that case. The linear combinations of columns of any matrix $\mathbf{A}$ are called the column space of $\mathbf{A}$. Project $\mathbf{b}$ onto the column space of $\mathbf{A}$ by solving $\mathbf{A}^T\mathbf{A}\hat{\mathbf{x}} = \mathbf{A}^T\mathbf{b}$ and $\mathbf{p} = \mathbf{A}\hat{\mathbf{x}}$. Find $\mathbf{e} = \mathbf{b} - \mathbf{p}$. Verify that it is perpendicular to $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

c) [Puk00] From the prequel we note that the matrix inversion $\left(\mathbf{A}^T\mathbf{A}\right)^{-1}$ might be computationally demanding. Since we are clever we choose the data $a_1, a_2, \ldots$ so that $\mathbf{A}^T\mathbf{A}$ becomes diagonal. Show that (P2) reduces to

$$\hat{\mathbf{h}}_{ls} = \frac{1}{P}\mathbf{A}^T\mathbf{y} \tag{P3}$$

if the transmitted data vector is

a=[-1 -1 1 -1 -1 1 -1 1 1 1 -1 -1 -1 -1 1 -1 -1 -1 1 -1 -1 1 -1 1 1 1]

and $L = 5$, $P = 16$.

Note: The data sequence $\mathbf{a}$ is one of the eight GSM training sequences and we shall use it throughout this laboratory work.

d) Let $\mathbf{e} \triangleq \mathbf{h} - \hat{\mathbf{h}}_{ls}$, where $\hat{\mathbf{h}}_{ls}$ is given by (P3). Assume that noise is zero-mean Gaussian with covariance matrix $\mathbf{C}_{nn} = E\left[\mathbf{nn}^T\right] = \sigma_n^2\mathbf{I}$. Show that the estimation error covariance matrix $\mathbf{C}_{ee} = E\left[\mathbf{ee}^T\right] = \frac{1}{P}\sigma_n^2\mathbf{I}$. Show also that the estimate is unbiased, i.e. $E\left[\hat{\mathbf{h}}_{ls}\right] = \mathbf{h}$. $E[\cdot]$ denotes the expectation operator.

Note: In part a you showed that $\hat{\mathbf{h}}_{ls} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{y}$ is the LS estimate of $\mathbf{h}$. In this part you actually showed that if $\mathbf{n}$ is a zero-mean, white Gaussian vector and $\mathbf{A}$ is chosen as in part c, $\hat{\mathbf{h}}_{ls}$ is also an efficient estimator of $\mathbf{h}$. In other words, in terms of variance, it is the best unbiased estimator of $\mathbf{h}$.

e) Optional problem for estimation gurus: An efficient estimator is one whose covariance satisfies the Cramer-Rao lower bound with equality. Find the CRLB for an unbiased estimate of $\mathbf{h}$ and verify that it equals $\frac{1}{P}\sigma_n^2\mathbf{I}$ thus showing that $\hat{\mathbf{h}}_{ls}$ is in fact an efficient estimator.

P2 (MATLAB implementation of channel estimator)

Implement a channel estimator using MATLAB. Your estimator should be able to handle a complex-valued QPSK modulated input signal and a complex-valued FIR channel. Channel noise is assumed to be complex WGN. Your estimator should be efficient in terms of variance and computation. Use the file `est.m` as a template for your MATLAB function. You can test your estimator function using the script `t_est.m`. Verify that the estimator is unbiased and its variance agrees with the result of problem P1d. Your estimator function will be used in the post-laboratory part.

P3 (Nyquist signaling, RC and RRC)

a) Create a BPSK modulated baseband signal using raised cosine (RC) signaling. Plot a segment of the signal and illustrate how intersymbol interference disappears if the signal is sampled at correct time instants. Produce another plot using root raised cosine signaling (RRC) signaling and illustrate that ISI does not disappear.

b) Filter a BPSK/RRC signal with another identical RRC filter and plot a segment of the output. Illustrate how ISI disappears. Relate the group delay of the cascade of two filters to the length of the impulse response of a single RRC filter. Plot and compare the impulse responses of a cascade of two RRC filters and an RC filter.

c) Produce scatter plots of sampled[2] BPSK/RC output using the correct sampling instant k, and sampling instants $k \pm 1$. Use four samples per symbol. Repeat for $\alpha = \{0, 0.5, 1\}$. α is the roll-off factor of the RC filter. Comment on the results. In your opinion, what is the effect of roll-off factor to receiver synchronization?

d) In practical communication systems RRC pulse shaping is always used instead of RC. Why?

P4 (QPSK transmitter/receiver for AWGN channel)

Simulate symbol error rate of QPSK modulation over an AWGN channel (Figure 1). Use RRC signaling with $\alpha = 0.5$ and four samples per symbol. Use the script `t_ser.m` as a template. Compare your simulation results to theory, see e.g. pages 271-273 of [Pro95] or any communications theory text book. The script will be later expanded to a receiver for bandlimited channel.

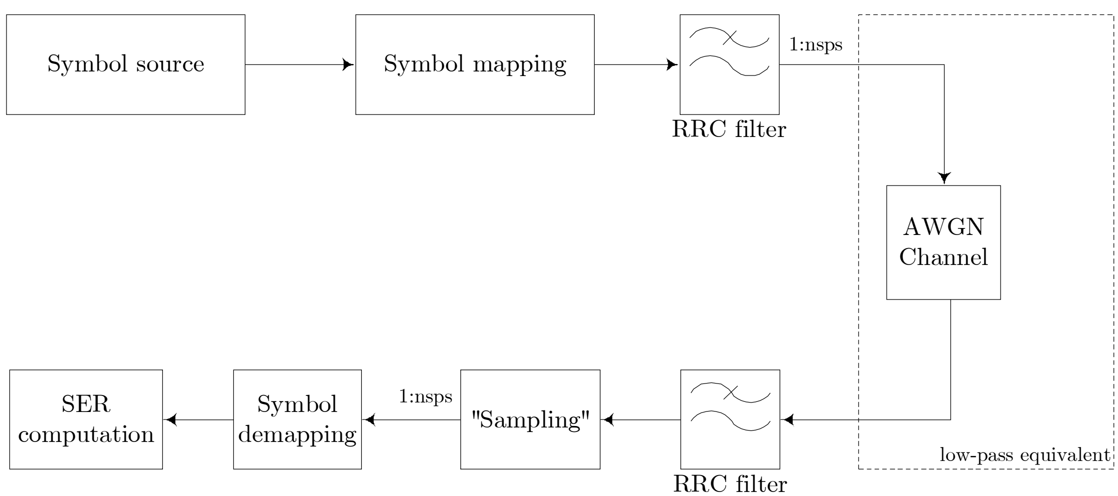

Figure 1. Communication system using RRC signaling.

[2] Since the data is already discrete-time a more accurate term would be decimation.

P5 (ZF equalizer)

Implement a zero-forcing equalizer using MATLAB. Your equalizer should work with arbitrary channel length and equalizer length; both are assumed to be odd, though. (Later it will have to work also with complex channel and complex signal.) Use the file `equ.m` as a template for your equalizer function. You can test your equalizer using the script `t_equ.m`. The function will be used later in the post-laboratory part as the equalizer block of your receiver.

a) Plot the impulse response of the cascade of the channel and the equalizer for both channel A and channel B given in the script `t_equ.m`. Compute ISI energy for both channels. Which channel is "worse" in terms of ISI canceling performance of the ZF equalizer?

b) Plot the frequency response of the channel, equalizer, and the cascade for the two channels given.

c) Consider the ZF equalizer of 11 taps' length used to equalize the channel B in the script `t_equ.m`. Compute the noise enhancement of this equalizer, i.e. the ratio of equalizer output SNR to input SNR. Give your answer in dB. Compare to noise enhancement of the ZF equalizer for channel A. Draw some conclusions on the noise enhancement and ISI cancellation properties of a ZF equalizer. Does the noise enhancement change if you increase the equalizer length, say… to 31 taps?