Receivers for interfering symbols

Signal space receiver

- Estimate transmitted signal $\mathbf{x}$ from received samples $\mathbf{y}$ using signal model

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

  - $\mathbf{H}$ is assumed to be known (by estimation).
- Linear receivers
  - use linear algebra to construct symbol estimates $\hat{x}_m$
  - bit decisions (soft or hard) made by quantizing $\hat{x}_m$
- non-linear receivers
  - use discreteness of $\mathbf{x}$ to construct the symbol estimates
  - iterative receivers: iterated (linear algebra + decsions)
  - approximative Maximum likelihood sequence estimators

MAP and ML

- The optimum receiver finds Maximum A Posteriori (MAP) probability:
  - M alternatives for $\mathbf{x}$: $\{\mathbf{x}_m\}_{m=1}^{M}$
  - MAP : most probable $\mathbf{x}_m$ is

$$\hat{\mathbf{x}} = \arg\max_m P(\mathbf{x}_m|\mathbf{y})$$

- Maximum Likelihood and MAP
  - Bayes rule: $P(\mathbf{x}_m|\mathbf{y}) = \frac{p(\mathbf{y}|\mathbf{x}_m)P(\mathbf{x}_m)}{p(\mathbf{y})}$
  - If all signals equiprobable: $P(\mathbf{x}_m) = 1/M$

$$\arg\max_m P(\mathbf{x}_m|\mathbf{y}) = \arg\max_m p(\mathbf{y}|\mathbf{x}_m)$$

  - ML: maximum likelihood $\mathbf{x}_m$ is

$$\hat{\mathbf{x}} = \arg\max_m p(\mathbf{y}|\mathbf{x}_m)$$

Maximum Likelihood Metric

- for vector signal model

$$\begin{aligned}
\mathbf{y} &= \mathbf{H}\mathbf{x} + \mathbf{n} \\
p(\mathbf{n}) &= \frac{1}{(\pi \mathrm{N}_0)^{N_\mathrm{r}}} \mathrm{e}^{-|\mathbf{n}|^2/\mathrm{N}_0} \\
p(\mathbf{y}|\mathbf{x},\mathbf{n}) &= \delta^{2N_\mathrm{r}}(\mathbf{y} - \mathbf{H}\mathbf{x} - \mathbf{n}) \\
p(\mathbf{y}|\mathbf{x}) &= \int_{C^{N_\mathrm{r}}} d^{2N_\mathrm{r}}n \; p(\mathbf{y}|\mathbf{x},\mathbf{n}) \; p(\mathbf{n}) \\
\Rightarrow p(\mathbf{y}|\mathbf{x}) &= \frac{1}{(\pi \mathrm{N}_0)^{N_\mathrm{r}}} \mathrm{e}^{-|\mathbf{y}-\mathbf{H}\mathbf{x}|^2/\mathrm{N}_0}
\end{aligned}$$

- ML detection metric: $\mathcal{M}_{\mathrm{ML}} = |\mathbf{y} - \mathbf{H}\mathbf{x}\;|^2$
- ML decision: $\arg\min_m \mathcal{M}_{\mathrm{ML}}(\mathbf{x}_m)$
  - *exhaustive search* over all $\mathbf{x}_m$
  - often prohibitively complex

Linear Receivers

- Set of linear equations $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$
  1. Solve esimate $\hat{\mathbf{x}}$ from this set using linear algebra
  2. decide symbol based on $\hat{\mathbf{x}}$
- Apply linear filter (matrix) $\mathbf{F}$ to $\mathbf{y}$
- rows of filter: $\mathbf{F}^{\mathrm{H}} = [\mathrm{f}_1 \ \mathrm{f}_2 \ \ldots \ \mathrm{f}_{N_{\mathrm{t}}}]$
- Decision metric decouples

$$\mathcal{M}_{\mathbf{F}} = |\mathbf{x} - \mathbf{F}\mathbf{y}|^2 = \sum_n |x_n - \mathrm{f}_n^{\mathrm{H}}\mathbf{y}|^2$$

  - $\hat{x}_n$ is the symbol closest to $\mathrm{f}_n^{\mathrm{H}}\mathbf{y}$

When is a linear receiver the optimum receiver?

1. If $\mathbf{H}$ singular, $\hat{\mathbf{x}}$ cannot be solved from $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$ even if $\mathbf{n}$ known
   $\Rightarrow$ linear filter may be optimal only if $\mathbf{H}$ non-singular
2. A receiver is optimal if decision metric is equivalent to ML metric:

$$a\mathcal{M}_{\mathbf{F}} = \mathcal{M}_{\mathrm{ML}}$$

   - for all possible $\mathbf{x}, \mathbf{y}, \mathbf{H}$
   - proportionality constant a may depend on $\mathbf{H}, \mathbf{y}$ but not on $\mathbf{x}$

$$\begin{aligned}
&\Rightarrow \quad \mathbf{F} = \mathbf{H}^{-1} \\
&\Rightarrow \quad a\,(\mathbf{H}\mathbf{x} - \mathbf{y})^{\mathrm{H}} \left(\mathbf{H}^{-1}\right)^{\mathrm{H}} \mathbf{H}^{-1} (\mathbf{H}\mathbf{x} - \mathbf{y}) = |\mathbf{H}\mathbf{x} - \mathbf{y}|^2 \\
&\Rightarrow \quad \mathbf{H}^{\mathrm{H}}\mathbf{H} = a\mathbf{I}
\end{aligned}$$

- Thus linear receiver is the optimum receiver only if the channel is orthogonal, $\mathbf{H}$ proportional to a unitary matrix.
- Orthogonal signaling, no interference.

Matched Filter

- Matched Filter (MF) is one of the simplest linear receivers
- combine coherently all samples according to the symbol of interest

$$\begin{aligned}
\mathbf{F} &= \left(\mathrm{diag}\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)\right)^{-1} \ \mathbf{H}^{\mathrm{H}} \\
\mathbf{f}_n &= \left(\mathbf{h}_n^{\mathrm{H}}\mathbf{h}_n\right)^{-1} \mathbf{h}_n
\end{aligned}$$

  - $\mathbf{h}_n$ is n:th column of $\mathbf{H}$
  - The inverses are just a scaling of the decision surfaces
- Example: 2×2 channel

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

  - MF estimate is

$$\hat{x}_1 = \frac{h_{11}^* y_1 + h_{21}^* y_2}{\left(|h_{11}|^2 + |h_{21}|^2\right)} = x_1 + \frac{\left(h_{11}^* h_{12} + h_{21}^* h_{22}\right) x_2}{\left(|h_{11}|^2 + |h_{21}|^2\right)} + \text{filtered noise}$$

MRC filter

- MRC is similar to MF, with additional reliability scaling
- Definition of Maximum Ratio Combining:
  - diversity branches combined coherently
  - combining weights selected to *maxmize post-combining SINR*
  - assuming noise + interference corrupting branches uncorrelated
- optimum MRC weights will be solved below

Optimum MRC weights I

- Rewrite signal model for receiving x_k

$$\mathbf{y} = \mathbf{h}_k \; x_k + \underbrace{\sum_{j \neq k} \mathbf{h}_j \; x_j + \mathbf{n}}_{\equiv \mathbf{i}}$$

- the covariance of noise + interference is

$$\mathrm{E}\left\{\mathbf{i} \; \mathbf{i}^{\mathrm{H}}\right\} = \sum_{j \neq k} \mathbf{h}_j \; \mathbf{h}_j^{\mathrm{H}} + \mathrm{N}_0 I$$

- for MRC $\mathbf{i}$ is approximated as uncorrelated interference:

$$\mathrm{E}\left\{\tilde{i}_m \; \tilde{i}_l^*\right\} = \left(\sum_{j \neq k} |h_{mj}|^2 + \mathrm{N}_0\right) \; \delta_{ml}$$

Optimum MRC weights II

- The filter is

$$\mathbf{f}_k = \mathbf{A}_k \; \mathbf{h}_k$$

where $\mathbf{A}_k = \mathrm{diag}[a_{k1} \; a_{k2} \; \ldots \; a_{kN_t}]$ is a diagonal matrix of real reliability weights
- signal power after filtering is

$$S_k = \left|\mathbf{h}_k^{\mathrm{H}} \mathbf{A}_k \mathbf{h}_k\right|^2 = \left(\sum_m a_{km} \; |h_{mk}|^2\right)^2$$

- the (Approximative) noise plus interference power is

$$\begin{aligned} I_k &= \mathbf{h}_k^{\mathrm{H}} \mathbf{A}_k \; \mathrm{E}\left\{\tilde{\mathbf{i}} \; \tilde{\mathbf{i}}^{\mathrm{H}}\right\} \mathbf{A}_k \mathbf{h}_k \\ &= \sum_m a_{km}^2 |h_{mk}|^2 \left(\sum_{j \neq k} |h_{mj}|^2 + \mathrm{N}_0\right) \end{aligned}$$

Optimum MRC weights III

- The set of reliability weights may be scaled with any number without changing SINR
- choose scale so that signal power $S_k = \mu^2$
  - minimize interference + noise power subject to constraint

$$\sum_m a_{km} \; |h_{mk}|^2 = \mu$$

  - Lagrangian optimization:

$$\mathcal{L} = I_k + 2\lambda\left(\mu - \sum_m a_{km} \; |h_{mk}|^2\right)$$

    - λ is a Lagrange multiplier

Optimum MRC weights IV

- Find minima of $\mathcal{L}$

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}a_{km}} = 2a_{km}|h_{mk}|^2\left(\sum_{j \neq k} |h_{mj}|^2 + \mathrm{N}_0\right) - 2\lambda|h_{mk}|^2 = 0$$

$$\Rightarrow \quad a_{km} = \frac{\lambda}{\sum_{j \neq k} |h_{mj}|^2 + \mathrm{N}_0}$$

- MRC weights are scaled by the interference + noise power per branch

MRC example

- 2 × 2 MF example above
- diversity branches

$$
\begin{aligned}
y_1 &= h_{11}\, x_1 + h_{12}\, x_2 + n_1 \\
y_2 &= h_{21}\, x_1 + h_{22}\, x_2 + n_2
\end{aligned}
$$

- MF
  - coherent combining with weights h_{11}^* and h_{21}^*
  - SIR for symbol x_1 is (omitting N_0)

$$\mathrm{SIR}_1 = \left| \frac{|h_{11}|^2 + |h_{21}|^2}{h_{11}^* h_{12} + h_{21}^* h_{22}} \right|^2$$

- this is not MRC optimum

MRC example II

- when receiving x_1, interference powers in y_1 and y_2 are

$$
\begin{aligned}
\mathrm{E}\left\{\tilde{i}_1\ \tilde{i}_1^*\right\} &= |h_{12}|^2 + \mathrm{N}_0 \\
\mathrm{E}\left\{\tilde{i}_2\ \tilde{i}_2^*\right\} &= |h_{22}|^2 + \mathrm{N}_0
\end{aligned}
$$

- choosing $\lambda = |h_{12}|^2 + \mathrm{N}_0$, MRC reliability weights are

$$
\begin{aligned}
a_{11} &= 1 \\
a_{12} &= \frac{|h_{12}|^2 + \mathrm{N}_0}{|h_{22}|^2 + \mathrm{N}_0} \equiv a
\end{aligned}
$$

- full MRC coherent combining weights h_{11}^* and $a\ h_{21}^*$:

$$\mathrm{SIR}_1 = \left| \frac{|h_{11}|^2 + a\ |h_{21}|^2}{h_{11}^* h_{12} + a\ h_{21}^* h_{22}} \right|^2$$

MRC example III

- for example if channel is near orthogonal

$$h_{11} = h_{22} = 10, \quad h_{12} = h_{21} = 1$$

- plain MF gives

$$\mathrm{SIR}_1^{\mathrm{MF}} = \left(\frac{101}{20}\right)^2 \approx 25$$

- MRC gives

$$\mathrm{SIR}_1^{\mathrm{MRC}} = \left(\frac{100 + a}{10 + 10a}\right)^2 = \left(\frac{100.01}{10.1}\right)^2 \approx 100$$

- NOTE: MRC does not maximize SINR
  - MRC: best SINR assuming *uncorrelated* noise + interference

MRC & MF terminology

- In textbooks, typically no distinction of the kind above is done between MRC & MF
  - MF/MRC for coloured noise is typically not treated
- A matched filter is by definition a filter which maximizes SINR assuming uncorrelated noise, be it white or coloured
  - see e.g. Benedetto-Biglieri, excercise 2.26, p. 102
- MF vs MRC as used here is slight misuse of terminology
  - in signal space, MF is by definition MRC, even with coloured noise
  - white-noise approximated MF or white-noise approximated MRC would be more accuate
- For conciseness of expression, MF and MRC as defined above will be used below

Zero Forcing

- Why not solve $\mathbf{x}$ directly form $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$, forgetting the noise?

$$\hat{\mathbf{x}} = \mathbf{H}^{-1}\mathbf{y}$$

- If $\mathbf{H}$ is singular, use Moore-Penrose pseudo-inverse:

$$\hat{\mathbf{x}} = \left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{H}}\ \mathbf{y}$$

  - note: if $\mathbf{H}$ non-singular, we have

$$\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{H}} = \mathbf{H}^{-1}\left(\mathbf{H}^{\mathrm{H}}\right)^{-1}\mathbf{H}^{\mathrm{H}} = \mathbf{H}^{-1}$$

- ZF symbol estimate

$$\hat{\mathbf{x}} = \mathbf{x} + \text{coloured noise}$$

  - all ISI has been forced to zero
  - noise is coloured (if channel not orthogonal)

Minimum Mean Square Estimator (MMSE)

First calculate some covariance matrices

$$\begin{aligned}
\mathrm{E}\left\{\mathbf{y}\mathbf{x}^{\mathrm{H}}\right\} &= \mathrm{E}\left\{(\mathbf{H}\mathbf{x}+\mathbf{n})\mathbf{x}^{\mathrm{H}}\right\} = \mathbf{H}\,\mathrm{E}\left\{\mathbf{x}\mathbf{x}^{\mathrm{H}}\right\} = \mathbf{H} \\
\mathrm{E}\left\{\mathbf{y}\mathbf{y}^{\mathrm{H}}\right\} &= \mathrm{E}\left\{(\mathbf{H}\mathbf{x}+\mathbf{n})\left(\mathbf{x}^{\mathrm{H}}\mathbf{H}^{\mathrm{H}}+\mathbf{n}^{\mathrm{H}}\right)\right\} \\
&= \mathbf{H}\,\mathrm{E}\left\{\mathbf{x}\mathbf{x}^{\mathrm{H}}\right\}\ \mathbf{H}^{\mathrm{H}} + \mathrm{E}\left\{\mathbf{n}\mathbf{n}^{\mathrm{H}}\right\} = \mathbf{H}\mathbf{H}^{\mathrm{H}} + \mathrm{N}_0\mathbf{I}
\end{aligned}$$

MMSE II

- The Mean Square Error

$$\begin{aligned}
\mathcal{E} &= \mathrm{E}\left\{|\mathbf{x}-\mathbf{F}\mathbf{y}|^2\right\} = \mathrm{Tr}\ \mathrm{E}\left\{(\mathbf{x}-\mathbf{F}\mathbf{y})\left(\mathbf{x}^{\mathrm{H}}-\mathbf{y}^{\mathrm{H}}\mathbf{F}^{\mathrm{H}}\right)\right\} \\
&= \mathrm{Tr}\left[\mathrm{E}\left\{\mathbf{x}\mathbf{x}^{\mathrm{H}}\right\} - 2\,\mathrm{Re}\left[\mathbf{F}\mathrm{E}\left\{\mathbf{y}\mathbf{x}^{\mathrm{H}}\right\}\right] + \mathbf{F}\ \mathrm{E}\left\{\mathbf{y}\mathbf{y}^{\mathrm{H}}\right\}\mathbf{F}^{\mathrm{H}}\right] \\
&= \mathrm{Tr}\left[\mathbf{I} - 2\,\mathrm{Re}\left[\mathbf{F}\mathbf{H}\right] + \mathbf{F}\ \left(\mathbf{H}\mathbf{H}^{\mathrm{H}}+\mathrm{N}_0\mathbf{I}\right)\mathbf{F}^{\mathrm{H}}\right]
\end{aligned}$$

- Find extrema by differentiating w.r.t. the elements of $\mathbf{F}$:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\mathbf{F}} = -2\mathbf{H}^{\mathrm{H}} + 2\mathbf{F}\ \left(\mathbf{H}\mathbf{H}^{\mathrm{H}}+\mathrm{N}_0\mathbf{I}\right)$$

- Solve for extremum filter matrix:

$$\mathbf{F}_{\mathrm{MMSE}} = \mathbf{H}^{\mathrm{H}}\left(\mathbf{H}\mathbf{H}^{\mathrm{H}}+\mathrm{N}_0\mathbf{I}\right)^{-1}$$

- this filter minimizes the MSE

MMSE III

- To see difference of MMSE and ZF, use matrix inversion lemma

$$\mathbf{V}\left(\mathbf{A}^{-1}+\mathbf{V}\mathbf{V}^{\mathrm{H}}\right)^{-1} = \left(\mathbf{I}+\mathbf{V}^{\mathrm{H}}\mathbf{A}\mathbf{V}\right)^{-1}\mathbf{V}^{\mathrm{H}}\mathbf{A}$$

- MMSE is ZF regularized by noise term:

$$\mathbf{F}_{\mathrm{MMSE}} = \left(\mathbf{H}^{\mathrm{H}}\mathbf{H}+\mathrm{N}_0\mathbf{I}\right)^{-1}\mathbf{H}^{\mathrm{H}}$$

  - For small N_0, MMSE becomes ZF
  - For large N_0, MMSE becomes MF (up to scaling)
  - ZF and MMSE are ML if channel orthogonal
- if non-white noise , $\mathrm{E}\left\{\mathbf{n}\mathbf{n}^{\mathrm{H}}\right\} = \mathbf{C}$:

$$\mathbf{F}_{\mathrm{MMSE}} = \mathbf{H}^{\mathrm{H}}\left(\mathbf{H}\mathbf{H}^{\mathrm{H}}+\mathbf{C}\right)^{-1} = \left(\mathbf{H}^{\mathrm{H}}\mathbf{C}^{-1}\mathbf{H}+\mathbf{I}\right)^{-1}\mathbf{H}^{\mathrm{H}}\mathbf{C}^{-1}$$

SINR analysis of linear receivers

SINR for generic linear receiver

- For performance analysis, post-processing SINR after linear receiver may be calculated
- possible residual ISI, and possibly coloured noise
- any linear receiver: $\mathbf{F}^{\mathrm{H}} = [\mathbf{f}_1\ \mathbf{f}_2\ \ldots\ \mathbf{f}_{N_\mathrm{t}}]$
- filter output for symbol k is

$$\begin{aligned} z_k &= \mathbf{f}_k^{\mathrm{H}}\ \mathbf{H}\ \mathbf{x}\ +\ \mathbf{f}_k^{\mathrm{H}}\ \mathbf{n} \\ &= \underbrace{\mathbf{f}_k^{\mathrm{H}}\ \mathbf{h}_k\ x_k}_{\text{wanted signal}}\ +\ \underbrace{\sum_{j \neq k} \mathbf{f}_k^{\mathrm{H}}\ \mathbf{h}_j\ x_j\ +\ \mathbf{f}_k^{\mathrm{H}} \mathbf{n}}_{\text{noise and interference}} \end{aligned}$$

  - Channel matrix is $\mathbf{H} = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \ldots & \mathbf{h}_{N_\mathrm{t}} \end{bmatrix}$

SINR for generic linear receiver II

- signal power after filtering is

$$S_k = \left|\mathbf{f}_k^{\mathrm{H}} \mathbf{h}_k\right|^2$$

- the noise plus interference power is

$$I_k = \sum_{j \neq k} \left|\mathbf{f}_k^{\mathrm{H}} \mathbf{h}_j\right|^2 + \mathrm{N}_0\ \mathbf{f}_k^{\mathrm{H}} \mathbf{f}_k$$

- post-processing SINR is

$$\mathrm{SINR}_k = \frac{S_k}{I_k}$$

MF, ZF, MMSE & Channel Covariance

- MF, ZF and MMSE can be written in the form: $\mathbf{F} = \mathbf{L}\mathbf{H}^{\mathrm{H}}$
  - $\mathbf{L}^{\mathrm{H}} = \begin{bmatrix} \mathbf{l}_1 & \mathbf{l}_2 & \ldots & \mathbf{l}_{N_\mathrm{t}} \end{bmatrix}$ is a channel inversion matrix
- filtered signal is

$$\mathbf{z} = \mathbf{L}\mathbf{R}\mathbf{x} + \mathbf{L}\mathbf{H}^{\mathrm{H}}\mathbf{n}\ ,$$

- the channel covariance matrix is

$$\mathbf{R} = \mathbf{H}^{\mathrm{H}}\mathbf{H}$$

  - diagonal elements: coherently combined (MF) channels of symbol x_k

$$r_{kk} = \left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)_{kk} = \mathbf{h}_k^{\mathrm{H}}\ \mathbf{h}_k$$

  - off-diagonal elements: ISI between x_k and x_j after MF

$$r_{kj} = \left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)_{kj} = \mathbf{h}_k^{\mathrm{H}}\ \mathbf{h}_j$$

Signal and interference power: MF, ZF, MMSE

- ★ Everything can be understood in terms of inversion matrix and channel covariance
- ★ Concentrate on symbol x_k
- ★ signal power after filtering

$$S_k = \left|\mathbf{l}_k^{\mathrm{H}} \mathbf{H}^{\mathrm{H}} \mathbf{h}_k\right|^2 = \left|(\mathbf{LR})_{kk}\right|^2$$

- ★ noise plus interference power is

$$\begin{aligned} I_k &= \sum_{j \neq k} \left|\mathbf{l}_k^{\mathrm{H}} \mathbf{H}^{\mathrm{H}} \mathbf{h}_j\right|^2 + \mathrm{N}_0\ \mathbf{l}_k^{\mathrm{H}} \mathbf{R}\ \mathbf{l}_k \\ &= \sum_{j \neq k} \left|(\mathbf{LR})_{kj}\right|^2 + \mathrm{N}_0 \left(\mathbf{LRL}^{\mathrm{H}}\right)_{kk} \end{aligned}$$

Post-processing SINR

- ★ Denote $\mathbf{G} = \mathbf{LR}$
- ★ post-processing SINR:

$$\mathrm{SINR}_k = \frac{|g_{kk}|^2}{\sum_{j \neq k} |g_{kj}|^2 + \mathrm{N}_0 \left(\mathbf{LRL}^{\mathrm{H}}\right)_{kk}}$$

  - ► first term in denominator comes from residual post-processing self-interference
  - ► second term is possibly enhanced and coloured noise

Orthogonal channel, again

- ★ $\mathbf{H}$ proportional to a unitary matrix
- ★ channel covariance proportional to identity, $\mathbf{R} = r\mathbf{I}$
  - ► $r_{kk} = r$ is the gain of the MRC combined channels
- ★ optimum inversion matrix proportional to identity, $\mathbf{L} = l\ \mathbf{I}$
- ★ SINR becomes

$$\mathrm{SINR}_k = \frac{(lr)^2}{\mathrm{N}_0 l^2\ r} = \frac{r}{\mathrm{N}_0}\ ,$$

  - ► no residual self-interference
  - ► no noise enhancement

SINR for Matched Filter

- ★ inversion matrix inverts just the coherently combined powers.
  - ► linear scaling, no effect on SINR, omitted here:

$$\mathbf{L} = \mathbf{I}$$

- ★ SINR becomes

$$\mathrm{SINR}_k = \frac{r_{kk}^2}{\sum_{j \neq k} |r_{kj}|^2 + \mathrm{N}_0 r_{kk}} = \frac{r_{kk}}{\sum_{j \neq k} |r_{kj}|^2 / r_{kk} + \mathrm{N}_0}$$

  - ► no self-interference is suppressed
  - ► noise is not enhanced

SINR for Zero Forcing

- inversion matrix inverts the channel covariance,

$$\mathbf{L} = \mathbf{R}^{-1}$$

- $\mathbf{G} = \mathbf{I}$
- SINR becomes

$$\mathrm{SINR}_k = \frac{1}{\mathrm{N}_0 \left(\mathbf{R}^{-1}\right)_{kk}}$$

  - self-interference vanishes completely
  - noise is enhanced
  - for an orthogonal channel we reproduce the result above

Noise Enhancment by ZF, Example I

- Example: 2×2 channel

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

- The channel covariance matrix is

$$\begin{aligned} \mathbf{R} &= \begin{bmatrix} |h_{11}|^2 + |h_{21}|^2 & h_{11}^* h_{12} + h_{21}^* h_{22} \\ h_{12}^* h_{11} + h_{22}^* h_{21} & |h_{12}|^2 + |h_{22}|^2 \end{bmatrix} \\ &= \begin{bmatrix} r_{11} & r_{12} \\ r_{12}^* & r_{22} \end{bmatrix} \end{aligned}$$

Noise Enhancment by ZF, Example II

- The determinant of $\mathbf{R}$ is

$$\det \mathbf{R} = r_{11} r_{22} - |r_{12}|^2$$

- the inverse of $\mathbf{R}$ is

$$\mathbf{R}^{-1} = \frac{1}{\det \mathbf{R}} \begin{bmatrix} r_{22} & r_{12}^* \\ r_{12} & r_{11} \end{bmatrix}$$

- the ZF SINR for symbol x_1 is

$$\mathrm{SINR}_1 = \frac{1}{\mathrm{N}_0 \left(R^{-1}\right)_{kk}} = \frac{\det \mathbf{R}}{\mathrm{N}_0\ r_{22}} = \frac{1}{\mathrm{N}_0} \left(r_{11} - \frac{|r_{12}|^2}{r_{22}} \right) \leq \frac{r_{11}}{\mathrm{N}_0}$$

  - equality only if $r_{12} = 0$, i.e. orthogonal channel
  - comparing to contribution of N_0 to SINR for MF, noise is enhanced by ZF receiver

SINR for MMSE

- inversion matrix inverts channel covariance up to regularization,

$$\mathbf{L} = (\mathbf{R} + \mathrm{N}_0 \mathbf{I})^{-1}$$

- expression for SINR non-transparent
  - both noise enhancement and some residual ISI
- In limit $\mathrm{N}_0 \to 0$, Zero Forcing result reproduced
- In limit $\mathrm{N}_0 \to \infty$, we have

$$\mathbf{L} \to \frac{1}{\mathrm{N}_0} \mathbf{I}$$

  - in expression for SINR, $1/\mathrm{N}_0$ factors cancel
  - MF result reproduced
- For orthogonal channel, we have

$$\begin{aligned} \mathbf{L} &= (r + \mathrm{N}_0)^{-1} \mathbf{I} \\ \mathbf{G} &= \frac{r}{r + \mathrm{N}_0} \mathbf{I} \end{aligned}$$

  - result for MF and ZF reproduced

Performance Analysis

- With the SINR values calculated above, performance of a detector can be analyzed
- For example, if QPSK is used, the BER of a symbol with SINR_k is
  - BER_k = Q(sqrt[SINR_k]) where
  - Q(x) = ½ Erfc(x/Sqrt[2])
- The average performance can be estimated by averaging the BERs

Performance Example of linear detectors:
Multiuser Detection for UL CDMA

UL CDMA

- There are U users simultaneously transmitting
- each user is using a spreading code c_u of length SF "chips"
  - SF is the spreading factor
  - The spreading code is interpreted as a column vector

$$\mathrm{c}_u = [c_{1u}\ c_{2u} \dots c_{SF,u}]^{\mathrm{T}}$$

  - elements of spreading code have norm 1
  - usually $c_{ju} \in \{1, \ -1\}$ or $c_{ju} \in \{1, \ -1, \ \mathrm{j}, \ -\mathrm{j}\}$
- the user is spreading the transmission of each symbol x_u over SF chips
  - example: $SF = 4$, one-tap channel h_u, transmitted symbol x_u
  - received signal from the transmission of user u (noise omitted):

$$\mathbf{y}_u = h_u \begin{bmatrix} c_{1u} \\ c_{2u} \\ c_{3u} \\ c_{4u} \end{bmatrix} x_u$$

UL CDMA, Power control

- Power Control (PC) is required in CDMA UL due to near-far effect
  - if no PC, signal from a user close to base station drowns signal of a far-away user below dynamic range of A/D converter
- Fast PC
  - attempts to follow fast fading
  - instantaneous received signal power of different users $\sim$ equal
- Slow PC
  - attempts to follow slow fading
  - mitigate shadowing and path loss
  - average received power of different users $\sim$ equal

UL CDMA, simplification for MUD analysis

- in WCDMA, UL is asynchronous
  - timing of spreading codes of different users is not synchronized
  - new symbol starts in different chip for different users
- the spreading sequences of different users are not orthogonal
  - spreading codes are pseudo-random sequences
  - good cross-correlation and auto-correlation properties
- to simplify analysis of effect of inter-user interference on UL CDMA with and without Multiuser Detection (MUD), we assume synchronous UL with non-orthogonal spreading codes

UL CDMA MUD, signal model

- signal model

$$\mathbf{y} = \underbrace{\begin{bmatrix} h_1 & 0 & \cdots & 0 \\ 0 & h_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & h_U \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_U \end{bmatrix}}_{\equiv \mathbf{H}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_U \end{bmatrix} + \mathbf{n}$$

- elements of covariance matrix $r_{uv} = h_u^* h_v \; \mathbf{c}_u^{\mathrm{H}} \mathbf{c}_v$
  - interference between users u and v if spreading codes not orthogonal
- when elements of spreading code normalized to 1:

$$\mathbf{c}_u^{\mathrm{H}} \mathbf{c}_u = SF$$

- coherently combined channel gain for user u is $r_{uu} = SF \; |h_u|^2$

CDMA SINR for Matched Filter receiver

- MF (= MRC!) for CDMA is the well-known RAKE receiver
- Matched filter SINR for user u is

$$\mathrm{SINR}_u = \frac{r_{uu}}{\sum_{v \neq u} |r_{vu}|^2 / r_{uu} + \mathrm{N}_0} = \frac{SF \; |h_u|^2}{\sum_{v \neq u} |h_v|^2 \left|\mathbf{c}_v^{\mathrm{H}} \mathbf{c}_u\right|^2 / SF + \mathrm{N}_0}$$

  - we see the *processing gain* = SF against noise and interference
    - wanted signal combines coherently
    - noise and interference non-coherently
- for random sequences $\mathrm{E}\left\{\mathbf{c}_u^{\mathrm{H}} \mathbf{c}_v\right\} = \sqrt{SF}$ for $u \neq v$
  - can be used to approximate SINR when many interferers
- with perfect PC

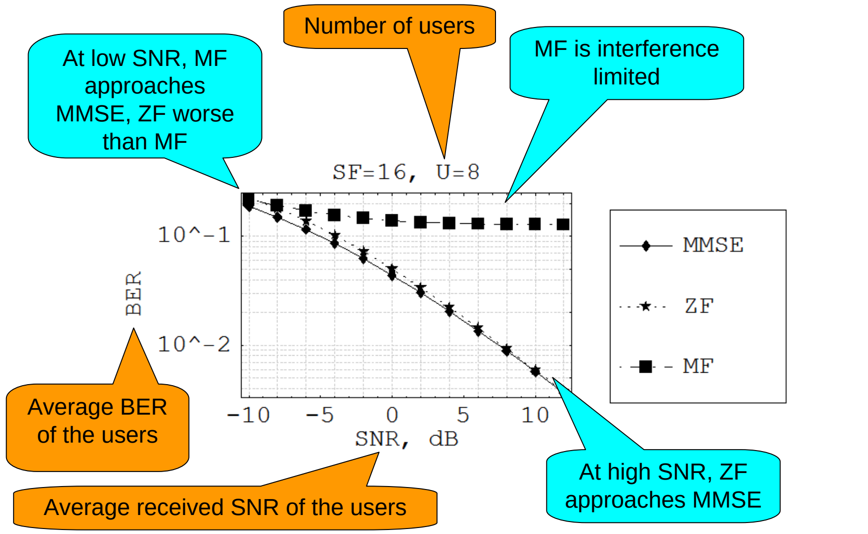
$$\mathrm{SINR}_u = \frac{SF}{U - 1 + \mathrm{N}_0 / |h_u|^2}$$

  - with increasing load, SINR decreases
  - with “full load”, $U = SF$, SINR ≈ 0 dB

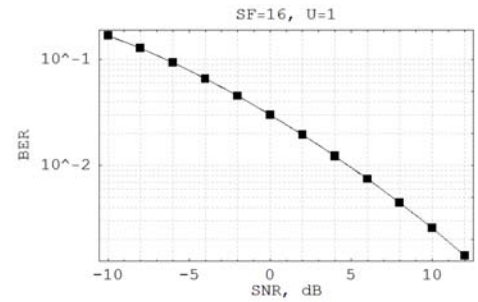
UL CDMA MUD Performance Plots

- next pages: performance plots
  - synchronous CDMA, $SF = 16$
    - processing gain $10 \log_{10}(16) = 12.04$ dB
  - random complex spreading codes
  - different number of users from $U = 1$ to $U = 16$
  - MF, ZF and MMSE receivers
  - slow and fast PC

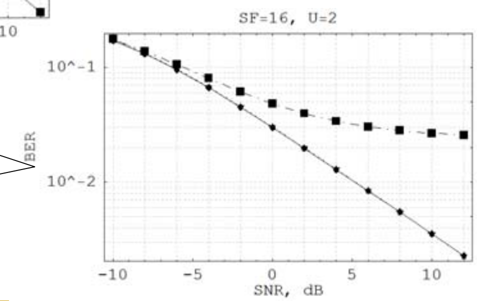
Typical plot



One & two users, slow PC

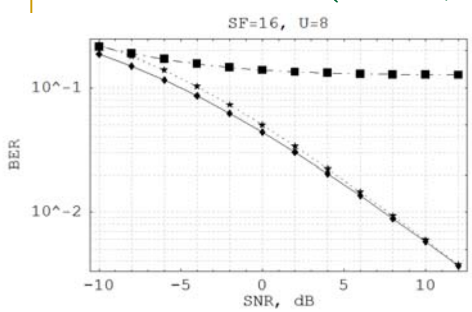


Single user: orthogonal system. All detectors same performance

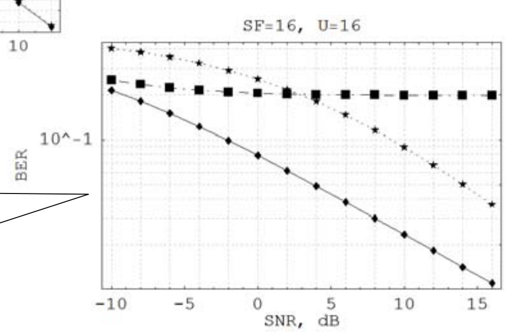


Adding a user makes MF interference limited. practically no effect on ZF& MF

Half and full load (8 users, 16users), slow PC

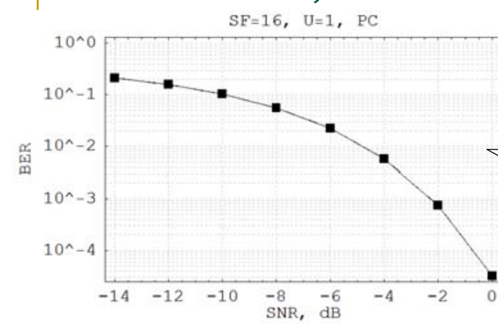


Half load:
MF error floor rises,
small effect on ZF & MMSE,
ZF at low SNR suboptimal

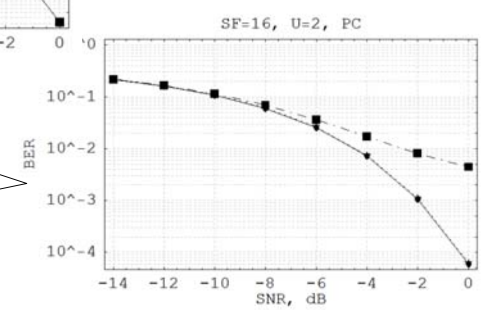


Full load:
MF error floor rises,
ZF very suboptimal,
4-6 dB loss to MMSE

One & two users, fast PC

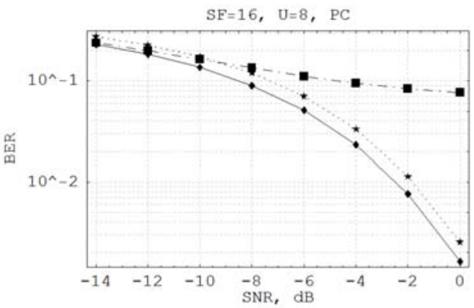


Single user: orthogonal system. Fast PC removed fading: AWGN performance in Rx SNR. Processing gain compared to AWGN ~12 dB

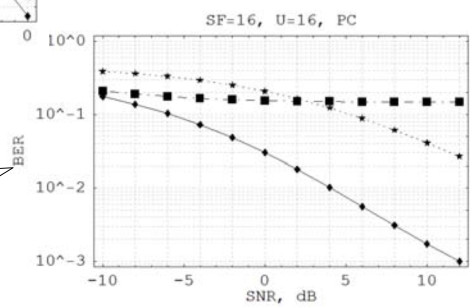


Adding a user makes MF interference limited. practically no effect on ZF& MF

Half and full load (8 users, 16users), fast PC



Half load:
MF error floor rises,
small effect on MMSE,
ZF > 1dB suboptimal



Full load:
MF error floor rises,
ZF 5-6 dB suboptimal

Observations from UL CDMA MUD

- ZF performs worst when system is most interference limited
  - this is a consequence of noise enhancement
  - when load is nearly full, channel covariance R has small eigenvalues
    - these lead to noise enhancement
  - regulating with N0 I in MMSE makes R better conditioned
    - less noise enhancement
- UL CDMA is desinged to operate with high load, close to ”pole capacity”
- at high load, ZF performs badly
- for MMSE, accurate estimate of N0, and R, required