Transversal Filters for ISI Channels

Toeplitz Matrix for ISI Channel

- ISI channel:

$$y_k = h_0\ x_k + \sum_{m=1}^{L-1} h_m\ x_{k-m}\ +\ n_k$$

- vector form

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

- with Toeplitz channel, example

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & 0 & 0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 & 0 & 0 \\ 0 & 0 & 0 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & 0 & 0 & h_2 & h_1 & h_0 \end{bmatrix} \begin{bmatrix} x_{-1} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{bmatrix}$$

Linear Equalizers for ISI: Transversal Filter

- construct a *transversal filter*
  - a set of taps operatng on a number of consecutive symbols, giving a symbol estimate
    - $\mathbf{f}_\mathrm{T}$ is $1 \times N_\mathrm{t}$ vector, $\mathbf{f}_\mathrm{T}^\mathrm{H}\mathbf{y} = \hat{x}_D$
  - the estimated symbol may be any of the symbols appearing in the received signals, given by delay D
    - in example above, it may be any of the $x_{-1}, \ldots, x_5$
    - the equlizer performance is different for different delay D
    - the best delay may be sought for (Krauss & al.)
    - often the middle symbol is taken, x_2 in example above

Transversal Filter III

- problem: $\mathbf{H}$ is $N_\mathrm{r} \times (N_\mathrm{r} + L - 1)$
  - more symbols in channel model than samples
  - $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$ is underdetermined
  - channel covariance $\mathbf{R}$ is singular
  - may be cured by oversampling (or multiple Rx antennas)
  - alternatively this may be simply overlooked
    - with MMSE, the matrix to be inverted is non-singular
    - poor performance for symbol at both ends
    - performance of symbols in the middle is almost optimum
    - with MRC, this is not a problem
- Example:
  - 4-tap channel, taps $[0.2,\ 0.9,\ 0.8,\ 0.2]$
  - SNR=10dB
  - 7-tap transversal filter, 7 samples, 7+4-1=10 symbols in filter
  - MMSE SINR for the symbols are:

$$[-12.72,\ 0.35,\ 1.64,\ 3.44,\ 3.95,\ 3.98,\ 3.87,\ 3.47,\ 3.42,\ -12.72]\mathrm{dB}$$

  - MRC SINR is -0.11 dB

Constructing Transversal Filter

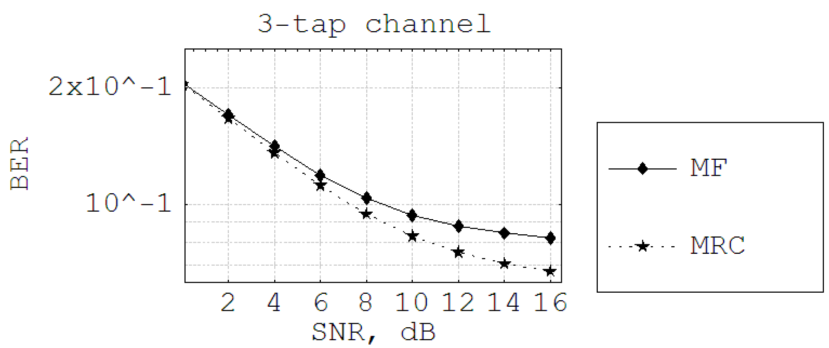
- ★ construct MMSE filter for vector channel model using Toeplitz matrix
- ★ either take filter column corresponding to central symbol $D = \lfloor N_t/2 \rfloor$,
- ★ or optimize delay D

$$\mathbf{f}_{\mathrm{T}} = \left(\mathbf{H}\left(\mathbf{H}^{\mathrm{H}}\mathbf{H} + \mathrm{N}_0 I\right)^{-1}\right)_D$$

ISI equalization & Transversal Filter performance

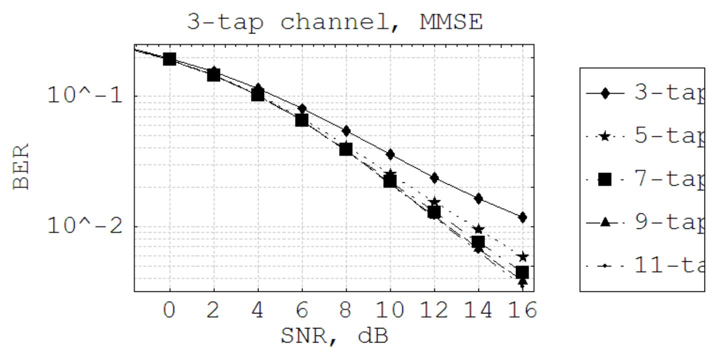
- 3-tap channel
  - tap amplitudes [0.41, 0.82, 0.41]
    - sum of tap powers normalized to 1
    - random phases for each tap
    - this set of taps includes some very difficult channels to equalize
      - the previous and next symbols may conspire to remove the center tap altogether, 0.41 x0 + 0.41 x2 may be – 0.82 x1
- 4-tap channel
  - tap amplitudes [0.15, 0.75, 0.6, 0.23]
- BER for QPSK modulation calculated
  - averaged over 1000 realizations of the tap phases

MRC vs MF, 3-tap channel



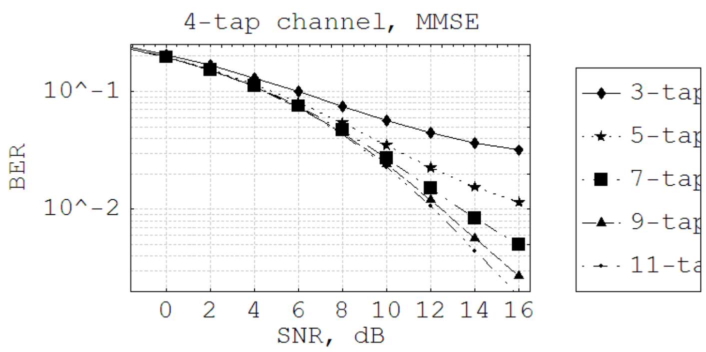
- true MRC several dB better at high SNR
- difference in BER not too significant
  - both suffer from error floor due to non-canceled interference

MMSE filter length, 3-tap channel



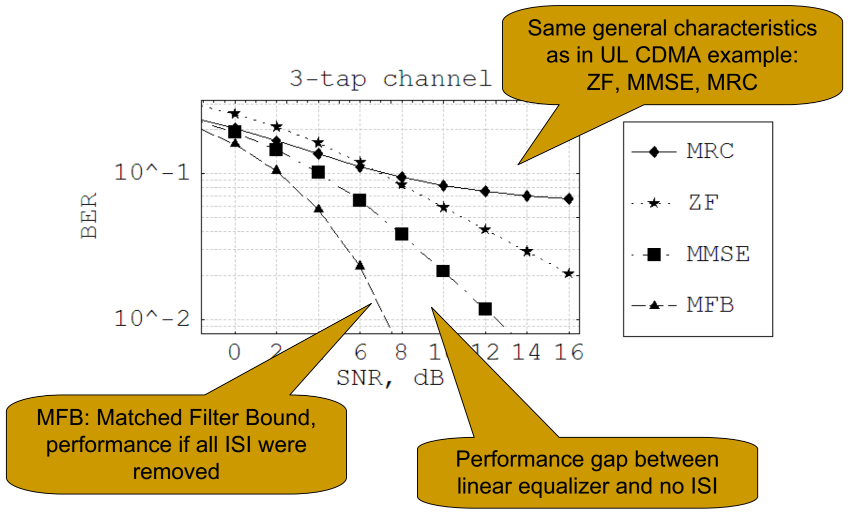
- performance improvement almost saturates at 7 taps

MMSE filter length, 4-tap channel



- performance improvement almost saturates at 9 taps
- usually 2L+1 taps is sufficient filter length

Performance of Different Transversal Filters



Summary: Transversal Filters

- A transversal filter estimates one symbol from N_r consecutive samples
- Filter taps solved in time domain
  - e.g. one column of MMSE filter matrix
  - 2L+1 taps typically enough
    - L taps to collect the energy of the symbol of interest
    - remaining taps to have sufficient independent samples to subtract interference
- the delay of the filter
  - which symbol is estimated from the set of samples
  - optimum delay depends on the Power Delay Profile
  - simple solution: take middle symbol

Frequency Domain Equalization, Block Transmission, and Cyclic Prefix

Circulant matrices, Diagonalization and Inversion

★ The rows of a circulant matrix are rotated versions of a basic row

$$\mathbf{C} = \begin{bmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{bmatrix}$$

★ a circulant matrix can be diagonalized with DFT:

$$\mathbf{MCM}^{\mathrm{H}} = \mathrm{diag}\begin{bmatrix} s_1 & s_2 & s_3 & s_4 \end{bmatrix} \equiv \Lambda$$

★ the $N \times N$ DFT matrix has elements $m_{kl} = \mathrm{e}^{2\pi \mathrm{j}\,(k-1)(l-1)/N}$

★ a circulant matrix is very simple to invert:

$$\mathbf{C}^{-1} = \mathbf{M}^{\mathrm{H}}\Lambda^{-1}\mathbf{M}$$

- ▶ Generic matrix inversion takes $O(N^3)$ multiplications
- ▶ with N a power of 2, invertion of circulant matrix takes $O(N \log_2 N)$ multiplications

Frequency Domain Equalization (FDE)

★ The channel covariance of a Toeplitz matrix is almost circular

★ Example: 3-tap channel, 5 Rx samples:

$$\mathbf{H} = \begin{bmatrix} h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & 0 & 0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 & 0 & 0 \\ 0 & 0 & 0 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & 0 & 0 & h_2 & h_1 & h_0 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} |h_2|^2 & h_2^* h_1 & r_2 & 0 & 0 & 0 & 0 \\ h_1^* h_2 & |h_1|^2 + |h_2|^2 & r_1 & r_2 & 0 & 0 & 0 \\ r_2^* & r_1^* & r_0 & r_1 & r_2 & 0 & 0 \\ 0 & r_2^* & r_1^* & r_0 & r_1 & r_2 & 0 \\ 0 & 0 & r_2^* & r_1^* & r_0 & r_1 & r_2 \\ 0 & 0 & 0 & r_2^* & r_1^* & |h_0|^2 + |h_1|^2 & h_1^* h_0 \\ 0 & 0 & 0 & 0 & r_2^* & h_0^* h_1 & |h_0|^2 \end{bmatrix}$$

$$\begin{aligned} r_0 &= |h_0|^2 + |h_1|^2 + |h_2|^2 \\ r_1 &= h_1^* h_0 + h_2^* h_1 \\ r_2 &= h_2^* h_0 \end{aligned}$$

FDE II

★ if the channel covariance is forced to circular, it can be inverted with DFT

- ▶ replace $\mathbf{R}$ with

$$\widetilde{\mathbf{R}} = \begin{bmatrix} r_0 & r_1 & r_2 & 0 & 0 & r_2^* & r_1^* \\ r_1^* & r_0 & r_1 & r_2 & 0 & 0 & r_2^* \\ r_2^* & r_1^* & r_0 & r_1 & r_2 & 0 & 0 \\ 0 & r_2^* & r_1^* & r_0 & r_1 & r_2 & 0 \\ 0 & 0 & r_2^* & r_1^* & r_0 & r_1 & r_2 \\ r_2 & 0 & 0 & r_2^* & r_1^* & r_0 & r_1 \\ r_1 & r_2 & 0 & 0 & r_2^* & r_1^* & r_0 \end{bmatrix}$$

- ▶ calculate $\widetilde{\Lambda} = \mathbf{M}\widetilde{\mathbf{R}}\mathbf{M}^{\mathrm{H}}$, the eigenvalue matrix of the approximative channel covariance $\widetilde{\mathbf{R}}$
- ▶ invert channel + noise covariance, calculate approximative MMSE weights:

$$\left(\mathbf{H}^{\mathrm{H}}\mathbf{H} + \mathrm{N}_0 I\right)^{-1} \mathbf{H}^{\mathrm{H}} \approx \mathbf{M}^{\mathrm{H}} \left(\widetilde{\Lambda} + \mathrm{N}_0 I\right)^{-1} \mathbf{M}\mathbf{H}^{\mathrm{H}}$$

Block Transmission and Guard Interval

★ transversal filter equalizes continuous single carrier transmission

- ▶ decision made on incomplete info of transmitted symbols
- ▶ some energy related to symbols in the equalization window always left outside the window
- ▶ all information available for equalization cannot be used except for infinite length equalizer

★ this can be improved by designing a **block transmission**

- ▶ a block of consecutive symbols is transmitted, preceded by a guard interval
  - ♦ length of GI at least $L-1$
- ▶ in the guard interval the ISI from the previous block is allowed to vanish, *no new information transmitted*
  - ♦ transmission rate lower for a block transmission than for continuous transmission

Block Transmission and GI: Example I

- guard interval filled with zeros

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & 0 & 0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 & 0 & 0 \\ 0 & 0 & 0 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & 0 & 0 & h_2 & h_1 & h_0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{bmatrix}$$

- samples $y_1, \ldots, y_5$, have only contributions from $x_1, \ldots, x_5$.
- no Inter-Block Interference
- symbols $x_1, \ldots, x_2$ equalized together from samples $y_1, \ldots, y_5$
- *lower rate* due to no information ni GI

Block Transmission and GI: Example II

- received power in *following* GI may be used to gather power for reception:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} h_0 & 0 & 0 & 0 & 0 \\ h_1 & h_0 & 0 & 0 & 0 \\ h_2 & h_1 & h_0 & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 \\ 0 & 0 & 0 & h_2 & h_1 \\ 0 & 0 & 0 & 0 & h_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \\ n_7 \end{bmatrix}$$

- the equalizer may now detect all $x_1, \ldots x_5$ from this block
- this requires calculating a full TDE filter for all symbols in the block
- FDE does not work
- after equalization, symbols in the block see a different channel
  - symbols close to block end suffer from less ISI

Cyclic Prefix

- A very special block transmission: fill GI with Cyclic Prefix (CP):

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & 0 & 0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 & 0 & 0 \\ 0 & 0 & 0 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & 0 & 0 & h_2 & h_1 & h_0 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{bmatrix}$$

$$= \begin{bmatrix} h_0 & 0 & 0 & h_2 & h_1 \\ h_1 & h_0 & 0 & 0 & h_2 \\ h_2 & h_1 & h_0 & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{bmatrix}$$

- channel matrix is circular!
- channel matrix and channel covariance can be diagonalized with DFT without loss
- after FDE-MMSE, all symbols see exactly the same channel, have same SINR
- Single-carrier transmisison with CP is UL modulation of LTE

OFDM I

- if the system has a CP, the channel is circulant

$$\mathbf{M}\mathbf{H}\mathbf{M}^{\mathrm{H}} = \Sigma$$

Σ is diagonal matrix of channel singular values
  - diagonal elements of Σ are the Fourier transform of first column of $\mathbf{H}$
  - i.e. Fourier transform of the Power Delay Profile
- channel can be diagonalized at transmitter

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} h_0 & 0 & 0 & h_2 & h_1 \\ h_1 & h_0 & 0 & 0 & h_2 \\ h_2 & h_1 & h_0 & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 \end{bmatrix} \mathbf{M}^H \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{bmatrix}$$

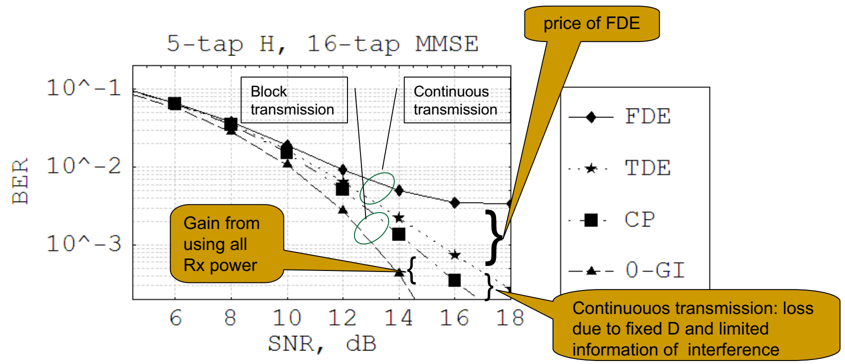
$$= \mathbf{M}^{\mathrm{H}}\Sigma\mathbf{x} + \mathbf{n}$$

  - requires no information of the channel at transmitter
  - makes channel orthogonal
  - removes all ISI
  - transmission moved to the frequency domain

OFDM II

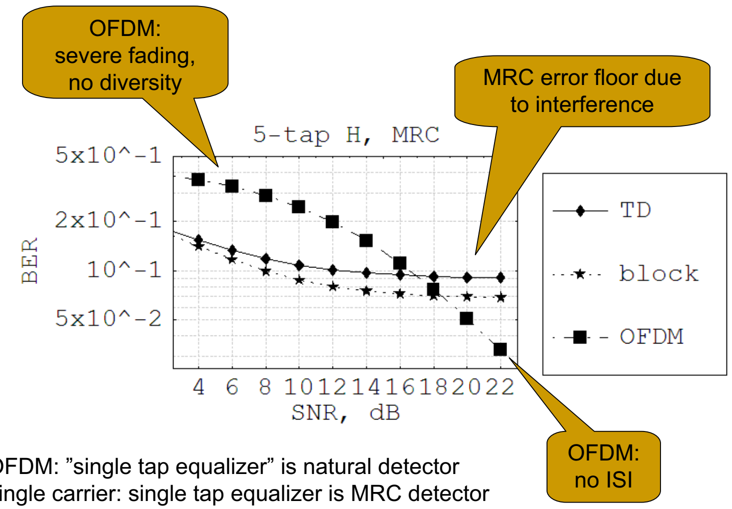
- This is Orthogonal Frequency Domain Multiplexing (OFDM)
- multicarrier transmission
  - there are N narrowband subcarriers
  - bandwidth W/N
  - a symbol is transmitted on a subcarrier
  - a subcarrier is a Fourier waveform
  - the narrowband channel observed on this subcarrier is Fourier transform of PDP
- receiver simply performs FFT (multiplying with $\mathbf{M}$)
- perfect removing of ISI with exceedingly simple receiver

16-tap MMSE, continuous & block transmissions



- FDE: frequency domain tap solving for continuous transmission
  - performance saturates at high SNR due to intentional error in channel inversion
- TDE: time domain equalization for continuous transmission
- CP: block transmission with cyclic prefix: FDE
- 0-GI: block transmission with silent guard interval. Block-TDE
  - note: block transmissions have lower rate

Single tap equalization: single carrier vs. OFDM



- OFDM: "single tap equalizer" is natural detector
- single carrier: single tap equalizer is MRC detector

Single tap equalization: single carrier vs. OFDM II

- uncoded BER: OFDM (much) worse than single carrier block transmission
  - block transmissions provide perfectly equalized multipath diversity
    - all symbols are received over the wide band, components from all cannel taps
  - OFDM has no ISI
    - narrowband transmission, no multipath diversity
    - frequency selective fading
  - recall that here the multiapath channel is not truly fading
    - amplitudes are fixed
- With diversity (though channel code etc) OFDM performance similar or slightly better than single carrier block transmission
  - slightly better with linear receivers: better channel estimation, no errors in equalization
  - here, channel estimation errors were not modeled

Summary: FDE

- with large equalizers, solving for transversal filter taps in time domain is computationally challenging
- approximative filter taps may be solved in frequency domain, using Fourier transform
  - approximation error decreases with length of filter, increases with length of channel
  - at high SNR; approximation error dominates performance
    - error floor due to intentional equalization inaccuracy

Summary Block transmissions & OFDM

- blocks of consecutive transmission symbols may be isolated from each other by inserting a Guard Interval between blocks
  - no inter-block interference
  - lower transmission rate due to GI
- equalization becomes a block-by-block operation
- If Guard Interval filled with Cyclic Prefix, FDE may be used without approximation
  - significant equalization complexity reduction when FFT applicable
- If CP is used, and pre-equalization with FFT is used at transmitter, we have constructed a multicarrier OFDM transmission
  - no ISI
  - each symbol sees a narrowband channel