MLSE in a single path channel

- MLSE - Maximum Likelihood Sequence Estimation
- The optimal detector is the one which selects from all possible transmitted bit sequences the one with highest probability
- In AWGN channel the sequence probability can be calculated as a multiplication of individual symbol probabilities

$$p\left(y_1, y_2, \ldots, y_N \mid x_1, x_2, \ldots, x_N\right) \overset{AWGN}{=} \prod_k p\left(y_k \mid x_1, x_2, \ldots, x_N\right)$$

$$\overset{Single\ path}{=} \prod_k p\left(y_k \mid x_k\right)$$

MLSE in a multipath channel

- A multipath channel mixes the symbols and therefore the probabilities of the symbols can not be calculated independently

$$p\left(y_1, y_2, \ldots, y_N \mid x_1, x_2, \ldots, x_N\right) \overset{\text{AWGN}}{=} \prod_k p\left(y_k \mid x_1, x_2, \ldots, x_N\right)$$

$$\overset{\text{L channel paths}}{=} \prod_k p\left(y_k \mid x_k, x_{k-1}, \ldots, x_{k-L-1}\right)$$

- For applying MLSE we have to know how to
  - create a state based model for the channel
  - calculate the conditional probability for the observed symbols

State model for a multipath channel

- The observed value y_k is modelled as a convolution of the channel impulse response h_{ch} and input symbols x observed in the noise n
  $y = h * x + n$
- In a L tap channel: to the value of y_k contribute only L previous symbols $x_k, x_{k-1}, \ldots, x_{k-L-1}$
- we define a state as a combination of $L-1$ previous bits

$$s_k = \begin{bmatrix} x_{k-1}, & x_{k-2}, & \ldots, & x_{k-L-1} \end{bmatrix}$$

- For a noise value with mean 0 the mean value of the sample y_k is defined by $f(x_k, s_k, h_{ch})$ and calculated as

$$f(x_k, s_k, h_{ch}) = h_1 \cdot x_k + h_2 \cdot x_{k-1} + \ldots h_L \cdot x_{k-L-1}$$

State model for a multipath channel

- The symbols $x_{k-1}, \ldots, x_{k-L-1}$ defining the state s can take values from the possible symbol set
  - For example for a BPSK modulation $x \in \pm 1$
  - Since each symbol could have two possible values we have in total 2^{L-1} possible states
  - At one stage the trellis will have 2^{L-1} possible states
- The x_k and the state s_k and h_{ch} define the new state s_{k+1}
  - The new state is combination of the new bit and $L-2$ youngest bit from the state s_k

$$\begin{aligned} s_{k+1} &= \begin{bmatrix} x_k & x_{k-1} & \ldots & x_{k-(L-2)} \end{bmatrix} \\ &= \begin{bmatrix} x_k & s_{k,\text{exluding}\, x_{k-L-1}} \end{bmatrix} \end{aligned}$$

Transition probabilities

- In a channel with only one path the observed value y_k is a function of only one symbol x_k $y_k = h_1 \cdot x_k + n$
- In a AWGN channel probability of the observed sample is

$$p(y_k|x_k, h_1) = \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(y_k - h_1 \cdot x_k)^2}{2\sigma_n^2}}$$

- In a multipath channel the mean value of the observed value y_k is computed based on the state s_k and the possible new bit x_k. The state transition probability is

$$p(y_k|x_k, s_k, h_{ch}) = \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(y_k - (h_1 \cdot x_k + h_2 \cdot x_{k-1} + \cdots + h_L \cdot x_{k-L-1}))^2}{2\sigma_n^2}}$$

Probabilty of a symbol sequence

- The probability of a possible transmitted symbols sequence calculated based on the observed symbols is

$$p(y_1, y_2, \ldots, y_n|x_1, x_2, \ldots, x_N, h_{ch}) = \prod_k p\left(y_k|x_1, x_2, \ldots, x_N, h_{ch}\right)$$
$$= \prod_k p\left(y_k|x_k, s_k, h_{ch}\right)$$

- We can calculate it in logarithmic domain

$$\log\left(p(y_1, y_2, \ldots, y_n|x_1, x_2, \ldots, x_N)\right) = \sum_k \log\left(p\left(y_k|x_k, s_k, h_{ch}\right)\right)$$
$$= \sum_k \left(\log\left(\frac{1}{\sqrt{2\pi\sigma_n^2}}\right) - \frac{1}{2\sigma_n^2}\left(y_k - f\left(x_k, s_k, h_{ch}\right)\right)^2\right)$$

Calculation of the transition probabilities ...

- The terms $\log\left(\frac{1}{\sqrt{2\pi\sigma_n^2}}\right)$ are common for all the sequences and we can drop them
- The scaling constant $-\frac{1}{2\sigma_n^2}$ is
  - the same for all the sequences: it scales all the sequences
  - since we want to find maximum over the sequences we can drop the scaling
  - since we dropped also multiplication by -1 the maximization becomes minimization

$$\log\left(p(y_1, y_2, \ldots, y_n|x_1, x_2, \ldots, x_N)\right) \Rightarrow \sum_k \left(\left(y_k - f\left(x_k, s_k, h_{ch}\right)\right)^2\right)$$

MLSE

- The maximum likelihood sequence estimation (MLSE) becomes a problem of finding the sequence with the minimum weight as following

$$\max_{\text{over all the sequences of } x_k} \left(p(y_1, y_2, \ldots, y_n|x_1, x_2, \ldots, x_N)\right)$$
$$\Rightarrow \min_{\text{over all the sequences of } x_k} \left(\sum_k \left(y_k - f\left(x_k, s_k, h_{ch}\right)\right)^2\right)$$

- Where

$$f\left(x_k, s_k, h_{ch}\right) = h_1 \cdot x_k + h_2 \cdot x_{k-1} + \cdots + h_L \cdot x_{k-(L-1)}$$

Viterbi algorithm

- The decoder has to find the bit sequence that generates the state sequence that is nearest to the received sequence y_k
- Each transition in the trellis depends only on the starting state and the end state
  - We know what would be the output from a state (without a noise)
  - The noiseless state output gives the mean value for the observation y_k
- The noise is Gaussian and we can calculate the probability based on the state transition (x_k, s_k, h_{ch}) and the received symbol y_k
- The total probability for a state sequence is the multiplication of all the probabilities along the path of the *"state sequence"* in the Markov chain

Viterbi algorithm...

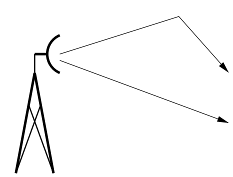
- For all transitions in the trellis compute the sum of the metrix in the initial state and in the transition
  - $L(s_k, s_{k+1}) = L(s_k) + L(y_k|s_k, s_{k-1})$
- At each state select among the incoming paths the one with the minimum metrix (the survival)
  - $L(s_{k+1}) = \min L(s_k, s_{k+1})$
- At each state remember the previous surviving state (in previous stage) and the bit corresponding to the surviving path

Example: Viterbi Algorithm

- Use a Viterbi decoder for estimating the transmitted bit sequence when the channel is $h_{ch} = 1 \cdot \delta(0) + 0.3 \cdot \delta(1)$.
- The receid signal is $y_{rec} = [1.3, 0.1, 1, -1.4, -0.9]$
- The input is a sequence of binary bits b modulated as x $1 \rightarrow 1$ $0 \rightarrow -1$.
- Estimate the most likely transmitted bit sequence

System description

- Two path channel generates two scaled copies of the transmitted bit sequence.
- Receiver sees sum of these copies plus noise



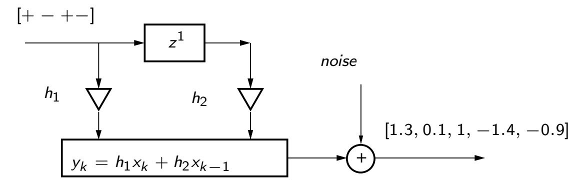
h_2		x_1	x_2	x_3	x_4
h_1	x_1	x_2	x_3	x_4	
			+		
	n_1	n_2	n_3	n_4	n_5
			=		
	y_1	y_2	y_3	y_4	y_5

$$y = h_{ch} * x + n$$

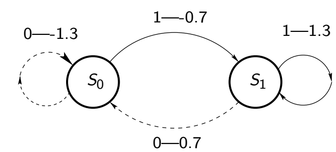
Construction of the Markov model

- The received sample y_k is a function of L transmitted symbols
- We can construct a state model where the state is defined by $L-1$ previous symbols and transition is defined by the new incoming symbol
  - L is the length of the channel response

Problem visualisation

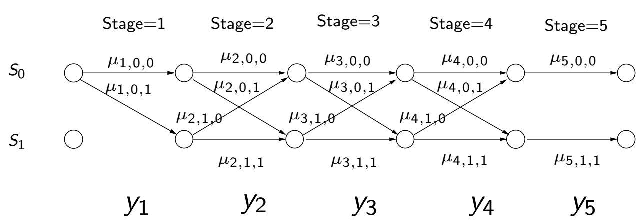


System state model for the trellis middle section

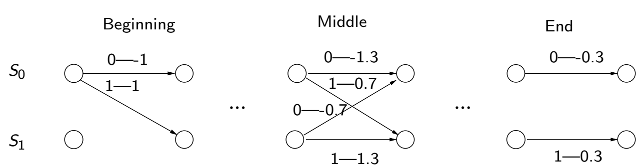


Trellis diagram construction

We assign costs $\mu_{Stage\ k,\text{initial state, final state}}$ for each transition



Transitions in different trellis sections

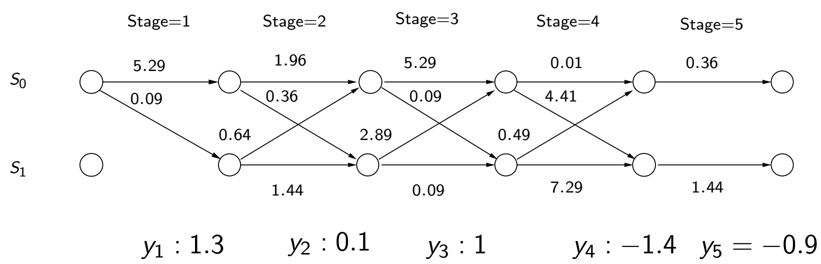


Transition calculation

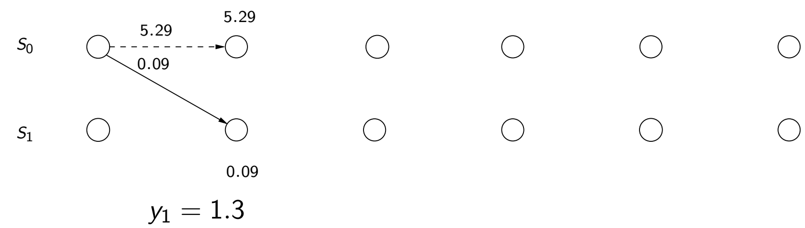
- Assign the weights for each transition
- At the trellis middle section
  - Normal opertional mode
- At the beginning of the trellis
  - Not all the channel taps are accounted
  - Last taps are not assigned any symbol since the signal along these paths have not yet arrived
- At the end of the trellis
  - The signal at the first taps have ended we have to account only the signal from the last channel taps

Example

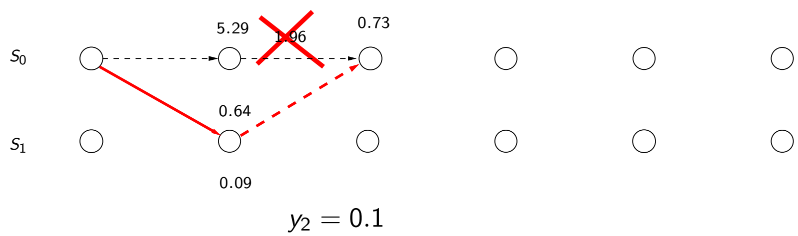
Trellis with the weights in transitions

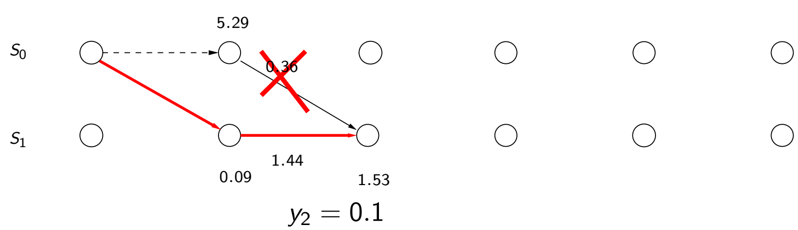


Decoding: Stage 1

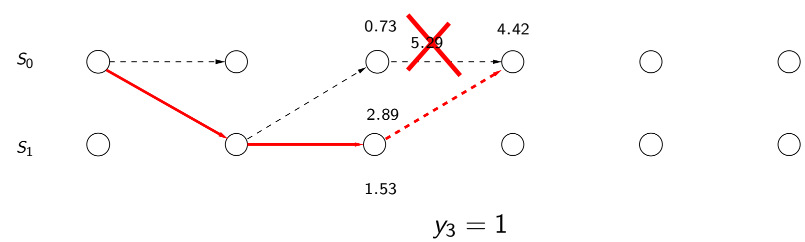


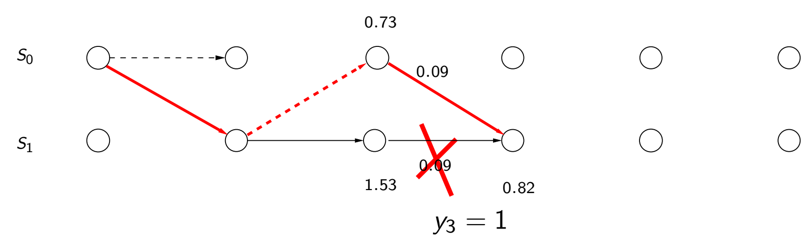
Decoding: Stage 2



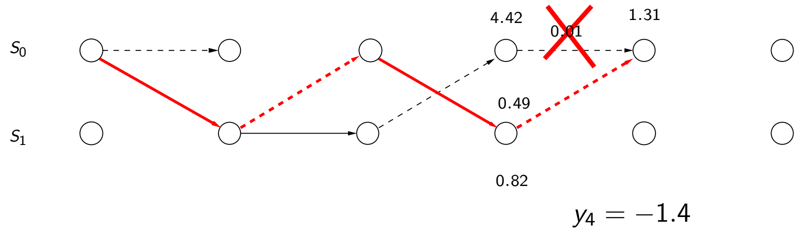


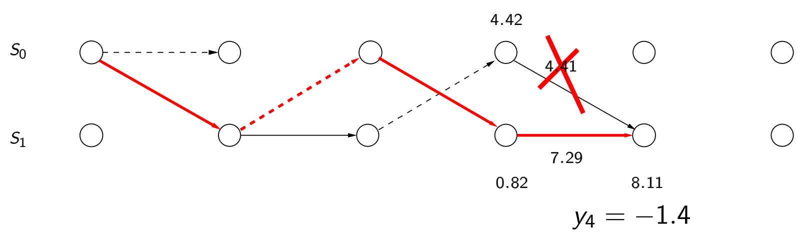
Decoding: Stage 3



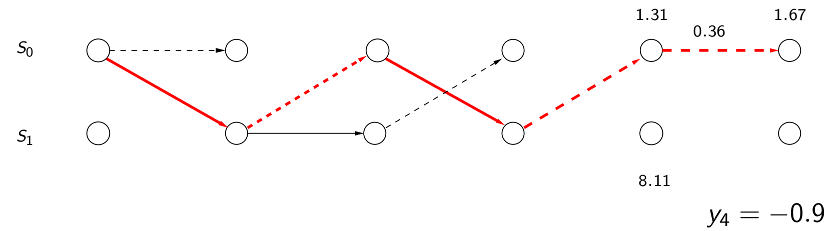


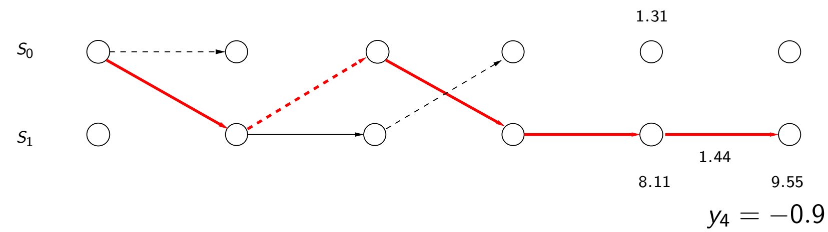
Decoding: Stage 4





Decoding: Stage 5





Example: Finding the ML sequence

- At the final *Stage* = 5 we have two states

$$S_0 \Rightarrow L(1.67)$$
$$S_1 \Rightarrow L(9.55)$$

- Select the one with minimum value and follow back along the remembered surving path
- The ML sequence is the bit sequence corresponding to this path. In our example it is 1 0 1 0 0

Example: Finding the ML sequence...

- The detected bit sequecne has 5 symbols the transmitted sequence has only 4 symbols
- One additional symbol, at the end of the sequence, is due to the convolution
  - Convolution of the symbol sequence with the channel has prolonged the sequence by $L-1$ elements
  - In our case $L=2$, and it gives 1 additional element
  - The additional symbols are the same as the last $L-1$ symbols