MAP algorithm

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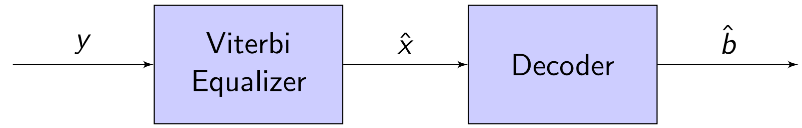
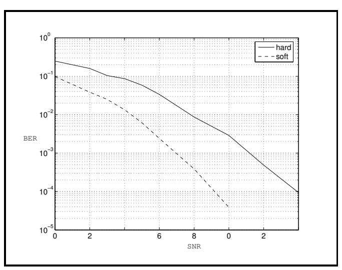
Need for a soft decoder



- Viterbi Equalizer provides only ML bit sequence $\hat{x}$
- ML sequence $\hat{x}$ contains hard bits
- The decoder following the Viterbi equalizer has to operate on hard bits

Performance



Comparison of soft and hard Viterbi decoder performance

- The performance of the decoder can be improved if the decoder operates on the soft values
  - Soft value contains not only the bit value but also the reliability information
  - For example soft value could be the bit probability

Soft bit calculation

- In order to have soft values at the input of the decoder the equalizer should generate a soft output
- Viterbi algorithm generates at the output only ML sequence $\rightarrow$ hard bits
- The soft information can be expressed as *aposteriori* probability of a bit
  - In the input to the equalizer the soft information is described as the probability of the input symbol
  - At the output the soft information can be expressed as *aposteriori* probability of the symbol
  - In case of binary bits the soft information can be expressed as loglikelihood ratio (llr)

Soft bit as loglikelihood ratio

- For deciding a binary bit value accordingly to Bayesian decision approach we have to calculate llr for a bit and compare it to the decision threshold
  - Loglikelihood ratio is a sufficient statistics - it contains all information for making optimal decision
  - In our setup (equalizer decoder) at the output of the equalizer we do not want to make decision (yet) but postpone it to the decoder stage
  - We provide into the decoder input sufficient statistics
- For bit sequence the llr for a bit has to be calculated by first marginalising the bit probabilities
  - Marignalization meas integrating (summing) over the values of nonrelevant variables

Marginalisation

- Marginalisation of a distribution of a variable is a removal of the impact of the nonrelevant variables
- For exmaple marginal distribution of a variable x_1 from the distribution of three variables x_1, x_2, x_3 is

$$p(x_1) = \iint_{-\infty}^{\infty} p(x_1, x_2, x_3) dx_1 dx_2$$

- for a discrete variables we could replace the integral over distribution by a sum over possible values

$$p(x_1) = \sum_{x_2} \sum_{x_3} p(x_1, x_2, x_3)$$

Marginalisation ...

- Further simpolification: if the variables are independent the probability can be split

$$p(x_1)p(x_2)p(x_3)$$

- The marginal proability of x_1

$$p(x_1) = p(x_1) \left(\sum_{x_2} p(x_2) \right) \left(\sum_{x_3} p(x_3) \right)$$

- Because we sum over probabilities: the sum over all possible values of x_2 and x_3 are 1

Soft bit

- LLR contain sufficient statistics for making optimal decision about the bit

$$\log \frac{p\,(x = 1|y)}{p\,(x = 0|y)}$$

- replace both of the conditional probabilities by using Bayes formula

$$p\,(x = 0|y) = \frac{p(y|x = 0)p(x)}{p(y|x = 0) + p(y|x = 1)} = \frac{p(y|x = 0)p(x)}{p(y)}$$

$$p\,(x = 1|y) = \frac{p(y|x = 1)p(x)}{p(y|x = 0) + p(y|x = 1)} = \frac{p(y|x = 1)p(x)}{p(y)}$$

- If the symbols are *apriori* equally probable we can write

$$\log \frac{p\,(x = 1|y)}{p\,(x = 0|y)} = \log\left(p\,(x = 1|y)\right) - \log\left(p\,(x = 1|y)\right)$$

Soft information from a symbol sequence

- Assume that the symbols are observed in the in additive white noise
- For a symbol sequence we have from marginalisation

$$\log\left(\frac{\sum_{x_k=1} p\left(x_1,x_2,\ldots,x_k=1,\ldots,x_N|y_1,y_2,\ldots,y_N\right)}{\sum_{x_k=0} p\left(x_1,x_2,\ldots,x_k=0,\ldots,x_N|y_1,y_2,\ldots,y_N\right)}\right)$$

- Based on Bayes formula

$$p\left(x_1,x_2,\ldots,x_k=1,\ldots,x_N|y_1,y_2,\ldots,y_N|\right) = \frac{p\left(y_1,y_2,\ldots,y_N|x_1,x_2,\ldots\ldots x_k=1\ldots,x_N\right)p\left(x_1,x_2,\ldots,x_k=1,\ldots,x_N\right)}{p\left(y_1,y_2,\ldots,y_N\right)}$$

$$p\left(x_1,x_2,\ldots,x_k=0,\ldots,x_N|y_1,y_2,\ldots,y_N|\right) = \frac{p\left(y_1,y_2,\ldots,y_N|x_1,x_2,\ldots,x_k=0,\ldots,x_N\right)p\left(x_1,x_2,\ldots,x_k=0,\ldots,x_N\right)}{p\left(y_1,y_2,\ldots,y_N\right)}$$

MAP algorithm

- We can simplify

$$\log\frac{\sum_{x_k=1} p\left(y_1,y_2,\ldots,y_k,\ldots,y_N|\,x_1,x_2,\ldots,x_k=1,\ldots,x_N\right)}{\sum_{x_k=0} p\left(y_1,y_2,\ldots,y_k,\ldots,y_N|\,x_1,x_2,\ldots,x_k=0,\ldots,x_N\right)}$$

$$\stackrel{AWGN}{=} \log\frac{\sum_{x_k=1}\left(\prod_{k_1} p\left(y_{k_1}|\,x_1,x_2,\ldots,x_k=1,\ldots,x_N\right)\right)}{\sum_{x_k=0}\left(\prod_{k_1} p\left(y_{k_1}|\,x_1,x_2,\ldots,x_k=0,\ldots,x_N\right)\right)}$$

$$\stackrel{single\ path}{=} \log\frac{\sum_{x_k=1}\left(\prod_{k_1} p\left(y_{k_1}|\,x_{k_1},x_k=1\right)\right)}{\sum_{x_k=0}\left(\prod_{k_1} p\left(y_{k_1}|\,x_{k_1},x_k=0\right)\right)}$$

MAP algorithm

$$\stackrel{multipath}{=} \log\frac{\sum_{x_k=1}\left(\prod_{k_1} p\left(y_{k_1}|\,x_1,x_2,\ldots,x_k=1,\ldots,x_N\right)\right)}{\sum_{x_k=0}\left(\prod_{k_1} p\left(y_{k_1}|\,x_1,x_2,\ldots,x_k=0,\ldots,x_N\right)\right)}$$

$$\stackrel{multipath\ Markov\ model}{=} \log\frac{\sum_{x_k=1}\left(\prod_{k_1} p\left(y_{k_1}|\,s_{k_1-1},x_{k_1},x_k=1\right)\right)}{\sum_{x_k=0}\left(\prod_{k_1} p\left(y_{k_1}|\,s_{k_1-1},x_{k_1},x_k=0\right)\right)}$$

MAP algorithm

- The last equation is the general form for the Maximum Aposteriori Probability (MAP) estimation algorithm
- The optimal algorithm for estimating not only bit values but also their reliability is maximum aposteriori probability (MAP) estimation for each bit in the received codeword
- For calculating the loglikelihood ratio from the *aposteriori* probabilities we have to sum over all possible bit sequences
  - Sum of the probabilities of the bit sequences where the bit $x_k = 1$ and where the bit $x_k = 0$
  - Take the logarithm of the ratio of these probabilities
- This is different compared to the Viterbi algorithm where we selected only one ML path
- MAP algorithm sums over all possible codewords (paths in the trellis)

Comparison of MAP and Viterbi algorithms

- Viterbi algorithm estiamtes the whole sequence, MAP algorithm estimates each bit separately
- Minimization of sequence (codeword) error does not mean that the bit error is minimized
- Viterbi algorithm estimates ML sequence
  - estimated $\hat{x}$ is a vector $\hat{\mathbf{x}}$ of hard bits corrsponding to ML sequence
  - the error probability is that given the received symbol vector $\mathbf{y}$ the estimted sequence $\hat{\mathbf{x}}$ is not equal to the transmitted codeword $\mathbf{x}$: $p(\hat{\mathbf{x}} \neq \mathbf{x}|\mathbf{y})$
  - Viterbi algorithm minimizes block error ratio (BLER)
  - no information about the reliability of the estimation

Comparison of MAP and Viterbi algorithms...

- MAP for each bit estimates the *a-posteriori* probability for each bit and minimizes bit error probability
  - $\hat{x}$ is a vector of likelihood ratios for each bit
  - the error probability is: given the received symbol vector $\mathbf{y}$ the estimations for each bit k $\hat{x}_k$ are not equal to the transmitted bit x_k: $p(\hat{x} \neq x|\mathbf{y})$
  - The loglikelihood ratio sign shows the bit value the amplitude describes reliability - how probable the decision is

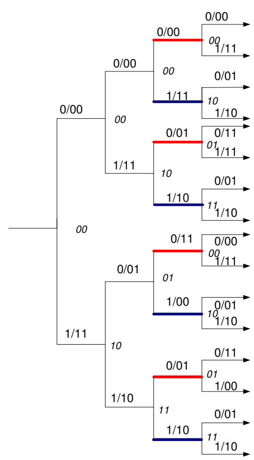
BCJR

- Implementation of MAP algorithm for decoding a linear code was proposed by Bahl, Cocke, Jalinek,and Raviv (BCJR) in 1974
  - For general trellis the algorithm is also known as *forward-backward* algorithm or Baum-Welch algorithm
- The algorithm is more complex than Viterbi algorithm
- When information bits are not equally likely the MAP decoder performs much better than Viterbi decoder
  - Used in iterative processing (turbo codes)
  - For turbo processing BCJR algorithm is slightly modified

Derivation of MAP decoding algorithm in trellis

- BCJR (MAP) algorithm for binary transmission finds the marginal probability that the received bit was 1 or 0
- Since the bit 1 (or 0) could occur in many different code words, we have to sum over the probabilities of all these code words
- The decision is made by using the likelihood ratio of these marginal distributions for $x = 1$ and $x = 0$
- The calculation can be structured by using trellis diagram
  - For every state sequence there is an unique path trough the trellis
  - Codewords sharing common state have common bit sequence. This sequence can be computed only once.
- The objective of the decoder is to examine states s and compute APPs associated with the state transitions

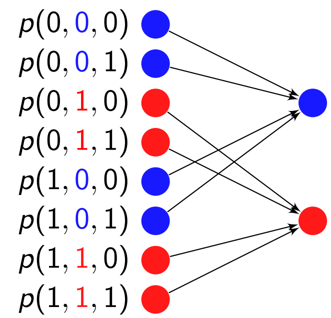
Marginal probability



- The probability of the code words is visualised in the code tree
- For independent bits the probability of one codeword is multiplication of probabilities of the individual bits in the codeword
- The marginal probability from the code tree for some particular bit beeing 1 or 0 corresponds to the sum of probabilities over all the codewords where this bit is 1 or 0
- A structured way for marginal probability calculation is to use trellis

Example: calculation of marginal probabilities

$p(c_1, c_2, c_3)$



$$p^{post}(c_2 = 0) = \frac{\sum_{c_2=0} p(c_1, c_2, c_3)}{\sum_{c_2=0} p(c_1, c_2, c_3) + \sum_{c_2=1} p(c_1, c_2, c_3)}$$

$$p^{post}(c_2 = 1) = \frac{\sum_{c_2=1} p(c_1, c_2, c_3)}{\sum_{c_2=0} p(c_1, c_2, c_3) + \sum_{c_2=1} p(c_1, c_2, c_3)}$$

Example: a-posteriori probability

For the independent samples we can separate

$$\begin{aligned} p^{post}(c_2 = 0) &= \frac{\sum_{c_2=0} p(c_1|c_2)p(c_2)p(c_3|c_1, c_2)}{\sum_{c_2=0} p(c_1|c_2)p(c_2)p(c_3|c_1, c_2) + \sum_{c_2=1} p(c_1|c_2)p(c_2)p(c_3|c_1, c_2)} \\ &= \frac{\sum_{c_2=0} p(c_1)p(c_2)p(c_3)}{\sum_{c_2=0} p(c_1)p(c_2)p(c_3) + \sum_{c_2=1} p(c_1)p(c_2)p(c_3)} \\ &= \frac{p(c_2 = 0)\left(\sum_{c_1} p(c_1)\right)\left(\sum_{c_3} p(c_3)\right)}{\sum_{c_1, c_3} p(c_1)p(c_3)\left(\sum_{c_2=0} p(c_1) + \sum_{c_2=1} p(c_2)\right)} \end{aligned}$$

Example:a-posteriori probability...

- The likelihood ratio becomes

$$\frac{p^{post}(c_2 = 1)}{p^{post}(c_2 = 0)} = \frac{p(c_2 = 1)\left(\sum_{c_1} p(c_1)\right)\left(\sum_{c_3} p(c_3)\right)}{p(c_2 = 0)\left(\sum_{c_1} p(c_1)\right)\left(\sum_{c_3} p(c_3)\right)}$$

$$= \frac{p(c_2 = 1) \cdot (p(c_1 = 0) + p(c_1 = 1)) \cdot (p(c_3) + p(c_3 = 0))}{p(c_2 = 0) \cdot (p(c_1 = 0) + p(c_1 = 1)) \cdot (p(c_3) + p(c_3 = 0))}$$

- The computation can be simplified by summing over all possible beginnings and ends of the codewords separately
  - In this simple example the sums can be reduced to 1, for a general code this is not the case

Derivation of the *forward-backward* algorithm

- For a general HMM the marginalisation can be simplified by grouping together the codewords that have a common terms.

$$\log \frac{\sum_{x_k=1}\left(\prod_{k_1} p\left(y_{k_1}\middle|\, s_{k_1-1}, x_{k_1}, x_k = 1\right)\right)}{\sum_{x_k=0}\left(\prod_{k_1} p\left(y_{k_1}\middle|\, s_{k_1-1}, x_{k_1}, x_k = 0\right)\right)}$$

- Let denote transition at stage k_1 from state $s_{k_1} = S_i$ to state $s_{k_1+1} = S_j$ in the next stage as

$$\begin{aligned} M_{k_1,j,i} &= p\left(y_{k_1} | s_{k_1} = S_i, s_{k_1+1} = S_j\right) \\ &= p\left(y_{k_1} | S_{k_1}, x_{k_1}\right) \end{aligned}$$

Derivation of the *forward-backward* algorithm

$$\sum_{x_k=1}\left(\prod_{k_1} p\left(y_{k_1}\middle|\, s_{k_1}, x_{k_1}, x_k = 1\right)\right) = \sum_{x_k=1}\left(\prod_{k_1} p\left(y_{k_1}\middle|\, s_{k_1}, s_{k_1+1}, x_k = 1\right)\right)$$

$\sum_{x_k=1}$ sum over all the symbol sequences where $x_k = 1$

$\prod_{k_1}$ multiplication over all the state transition along a path corresponding to some particular codeword

- We can regroup:
  - all the codewords contain one common transition $p(y_k | s_k, s_{k+1}, x_k = 1)$

Derivation of the *forward-backward* algorithm

$$\sum_{s_k \in S} p\left(y_k | s_k, s_{k+1}, x_k = 1\right) \cdot \left(\prod_{k_1<k} p\left(y_{k_1}\middle|\, s_{k_1}, s_{k_1+1}\right)\right) \cdot \left(\prod_{k_1>k} p\left(y_{k_1}\middle|\, s_{k_1}, s_{k_1+1}\right)\right)$$

$A_{k,i} = \left(\prod_{k_1<k} p\left(y_{k_1}\middle|\, s_{k_1}, s_{k_1+1}\right)\right)$ is called forward metrics

$B_{k,j} = \left(\prod_{k_1>k} p\left(y_{k_1}\middle|\, s_{k_1}, s_{k_1+1}\right)\right)$ is called backward metrics

- $\alpha_{k,i} = \log(A_{k,i})$ is sum of the proabilities along all the paths that when starting from the beginning of the trellis at the stage k will merge to state i
- $\beta_{k,j} = \log(B_{k,j})$ is sum of the proabilities along all the paths that when starting from the end of the trellis at the stage $k+1$ merge to state j

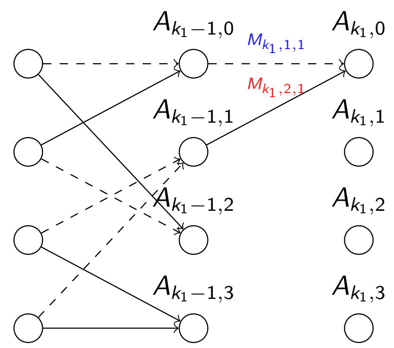
Calculation of metrics

- A and B can be calculated recursively
- for particular state k_i, i we can write corresponding $A_{k_1,i}$
  $A_{k_1,i} = \sum_{s_k=S_i} p(y_{k_1-1}, y_{k_1-2}, \ldots, y_1 | s_{k_1} = S_i, s_{k-1}, \ldots, s_1)$

$$\begin{aligned} A_{k_1,i} &= \sum_{s_{k_1-1}=S} \begin{pmatrix} p(y_{k_1-1} | s_{k_1} = S_i, s_{k_1-1} = S_{i_1}) \\ \cdot p(y_{k_1-2}, \ldots, y_1 | s_{k_1-1} = S_{i_1}, \ldots, s_1) \end{pmatrix} \\ &= \sum_{i_1} M_{k_1,i,i_1} \cdot A_{k_1-1,i_1} \end{aligned}$$

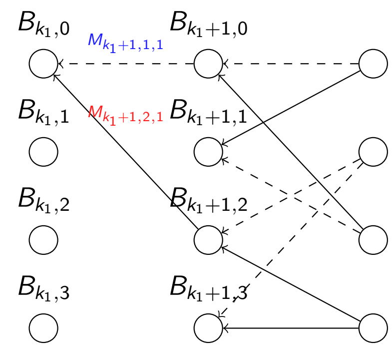
$$\begin{aligned} B_{k_1,j} &= \sum_{s_{k_1+1}=S} \begin{pmatrix} p(y_{k_1+1} | s_{k_1} = S_j, s_{k_1+1} = S_{j_1}) \\ \cdot p(y_{k_1+2}, \ldots, y_N | s_{k_1+1} = S_{j_1}, \ldots, s_N) \end{pmatrix} \\ &= \sum_{j_1} M_{k_1,j,j_1} \cdot B_{k_1+1,j_1} \end{aligned}$$

Illustration: froward metrics



$$\begin{aligned} A_{k_1,i} &= \sum_{i_1} M_{k_1,i,i_1} \cdot A_{k_1-1,i_1} \\ &= M_{k_1,1,1} \cdot A_{k_1-1,1} + M_{k_1,2,1} \cdot A_{k_1-1,2} \end{aligned}$$

Illustration: bacward metrics



$$\begin{aligned} B_{k_1,i} &= \sum_{i_1} M_{k_1+1,i,i_1} \cdot B_{k_1+1,i_1} \\ &= M_{k_1+1,1,1} \cdot B_{k_1-1,1} + M_{k_1+1+1,2,1} \cdot B_{k_1+1,2} \end{aligned}$$

Calculation in log domain

- For numerical stability better to compute in logarithmic domain

$$\begin{aligned} \log\left(A_{k,i} \cdot M_{k,i,j} \cdot B_{k,j}\right) &= \log\left(A_{k,i}\right) + \log\left(M_{k,i,j}\right) + \log\left(B_{k,j}\right) \\ &= \alpha_{k,i} + \mu_{k,i,j} + \beta_{k,j} \end{aligned}$$

$$\alpha_{k,i} = \log\left(A_{k,i}\right) \quad \beta_{k,j} = \log\left(B_{k,j}\right) \quad \mu_{k,i,j} = \log\left(M_{k,i,j}\right)$$

$$\begin{aligned} A_{k_1,i} &= \sum_{i_1} M_{k_1,i,i_1} \cdot A_{k_1-1,i_1} &\Rightarrow \quad \log\left(\sum_{i_1} e^{\mu_{k_1,i,i_1}+\alpha_{k_1-1,i_1}}\right) \\ B_{k_1,j} &= \sum_{j_1} M_{k_1+1+1,j,j_1} \cdot B_{k_1+1,j_1} &\Rightarrow \quad \log\left(\sum_{j_1} e^{\mu_{k_1+1,j,j_1}+\beta_{k_1+1,j_1}}\right) \end{aligned}$$

Metric for AWGN channel

- Probability calcualtion simplification for the equalizer in AWGN channel

$$\frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(y_k - f(x_k,s_k,h_{ch}))^2}{2\sigma_N^2}} \Rightarrow -\log\sqrt{2\pi\sigma_n^2} - \frac{(y_k - f(x_k,s_k,h_{ch}))^2}{2\sigma_N^2}$$

$$\Rightarrow -\frac{1}{2\sigma_N^2}\left(y_k^2 - 2y_k f(x_k,s_k,h_{ch}) + f(x_k,s_k,h_{ch})^2\right)$$

$$\Rightarrow \frac{1}{2\sigma_N^2} 2y_k f(x_k,s_k,h_{ch}) - \frac{1}{2\sigma_N^2} f(x_k,s_k,h_{ch})^2$$

Initialization

- Suppose the decoder starts and ends with known states.

$$A_{0,0} = \begin{cases} 1, & s = S_0 \\ 0, & \text{otherwise} \end{cases}$$

- If the final state is known

$$B_{N,0} = \begin{cases} 1, & s_N = S_0 \\ 0, & \text{otherwise} \end{cases}$$

- If the final state of the trellis is unknown

$$B_{N,j} = \frac{1}{2^m}, \ \forall s_N$$

Metric calculation

Map algorithm: Summing over all the codewords where the symbol $x_k = 0$ and where it is $x_k = 1$

- at the node we combine the probabilities meging into this node
  $\alpha_k(s_k = S_i) = \sum_{s_k=S_i} p(y_{k-1}, y_{k-2}, \ldots, y_1 | s_k = S_i, s_{k-1}, \ldots, s_1)$
- We do not reject other path but sum the probabilities togeher
- the probability of the part of the codeword continuing from the trellis state where they merge will be same for both codewords
- Similarly we can calculate for backward metric B by starting at the end of the trellis

Algorithm

- Initialize the forward and backward metrics $\alpha_0^*(k)$
- Compute the brance metrics $\mu_{k,i,j}$
- Compute the forward metrics $\alpha_{k,i}$
- Compute the backward metrics $\beta_{k,j}$
- Compute APP L-values $L(b)$
- (optional) Compute hard decisions

Example:MAP equalizer

- Use MAP bit estimation for estimating the transmitted bits if the the channel is $h_{ch} = 1 \cdot \delta(0) + 0.3 \cdot \delta(1)$.
- The received signal is $y_{rec} = [1.3, 0.1, 1, -1.4, -0.9]$
- The input is a sequence of binary bits b modulated as x $1 \rightarrow 1$ $0 \rightarrow -1$.
- The $EbN0 = 2dB$
- Estimate the most likely transmitted bits
- (Notice that we are using the same values as in the example about Viterbi equalizer.)

Metric assignment

- Metric is calculated based on the received value and the value in trellis branch.

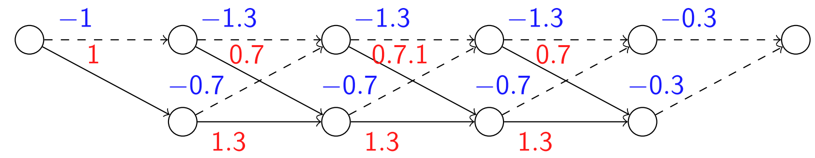
$$\frac{1}{||h_{ch}||^2}\left(\frac{1}{2\sigma_N^2}2y_k f(x_k, s_k, h_{ch}) - \frac{1}{2\sigma_N^2} f(x_k, s_k, h_{ch})^2\right)$$

where

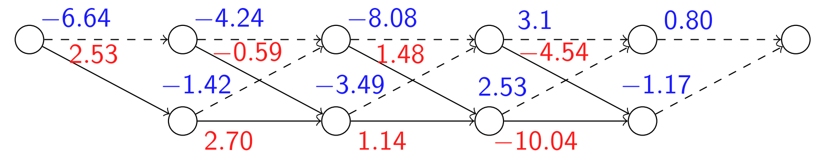
$$f(x_k, s_k, h_{ch}) = h_1 \cdot x_k + h_2 \cdot x_{k-1} + \cdots + h_L \cdot x_{k-(L-1)}$$

is the noiseless (mean) value of the corresponding trellis branch.
This metric is normilized by the channel total power $\frac{1}{||h_{ch}||^2}$

Metric on each branch



Mean value on each branch.

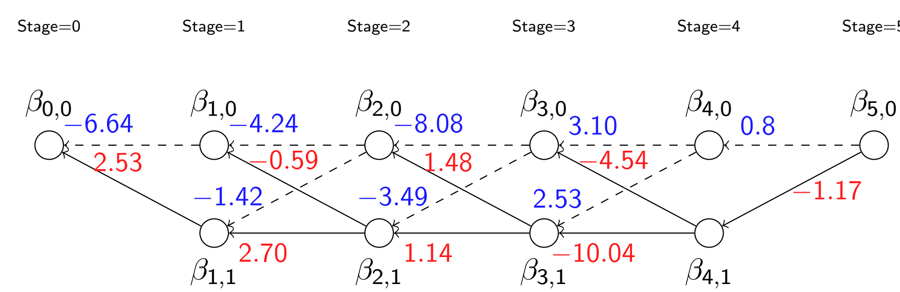


Metric in each trellis branch

Example: Backward calculation, Bacward trellis

- start from the end of the trellis and calcualate the sum of the probabilitites
- Init the last stage probability to 1 (in log domain to 0): $\beta_{5,0} = 0$
  1. By using the state probabilities $k+1$ and the probabilities on the transitions calculate the probabilities for the states k $\beta_{k,i} = \log\left(\sum_j e^{\mu_{k,i,j}+\beta_{k+1,j}}\right)$ (If the values are in the log domain for the sum you have to convert them back to the probability domain.)
  2. Normilize (needed for the numerical stability) $\beta_{k,\#} = \beta_{k,\#} - \max(\beta_{k,\#})$

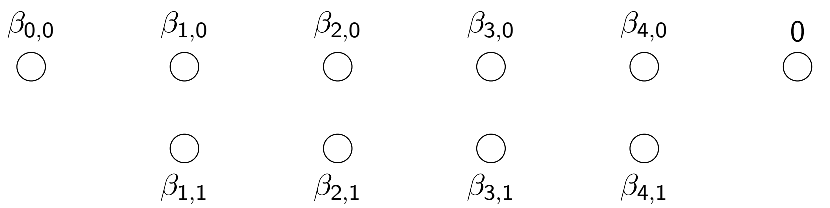
Example: Backward calculation, Bacward trellis



$\beta_{stage,state}$

$\beta_{k,i} = \log\left(\sum_j e^{\mu_{k+1,i,j}+\beta_{k+1,j}}\right)$

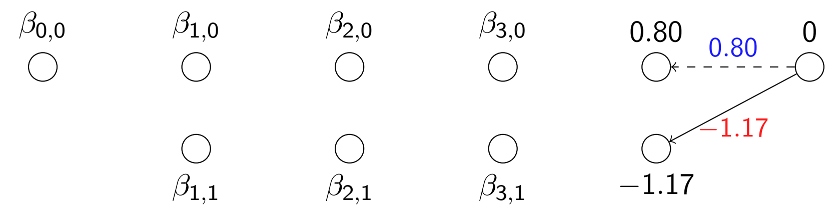
Example: Backward calculation: Stage 5 to 4



1. The new state probabilities
   $\beta_{4,0} = \mu_{5,0,0} + \beta_{5,0} = 0.80 + 0$
   $\beta_{4,1} = \mu_{5,1,0} + \beta_{5,0} = -1.17 + 0$
2. $\beta_{4,0} = 0.80 - (0.80) = 0,\ \beta_{4,1} = -1.17 - (0.80) = -1.97$

MAP algorithm
Example:Solution
Backward calculation

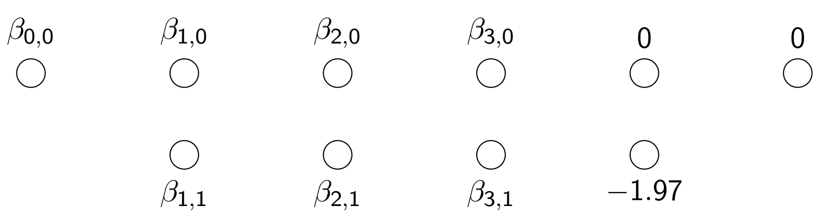
Example: Backward calculation: Stage 5 to 4



1. The new state probabilities
   $\beta_{4,0} = \mu_{5,0,0} + \beta_{5,0} = 0.80 + 0$
   $\beta_{4,1} = \mu_{5,1,0} + \beta_{5,0} = -1.17 + 0$
2. $\beta_{4,0} = 0.80 - (0.80) = 0,\ \beta_{4,1} = -1.17 - (0.80) = -1.97$

MAP algorithm
Example:Solution
Backward calculation

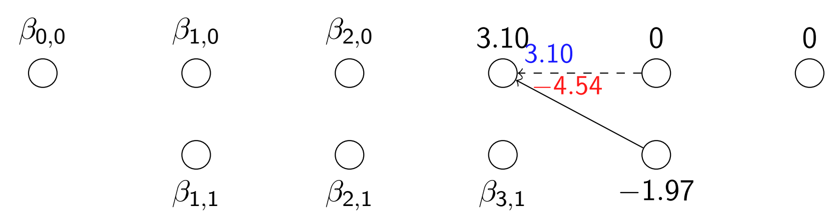
Example: Backward calculation: Stage 5 to 4



1. The new state probabilities
   $\beta_{4,0} = \mu_{5,0,0} + \beta_{5,0} = 0.80 + 0$
   $\beta_{4,1} = \mu_{5,1,0} + \beta_{5,0} = -1.17 + 0$
2. $\beta_{4,0} = 0.80 - (0.80) = 0,\ \beta_{4,1} = -1.17 - (0.80) = -1.97$

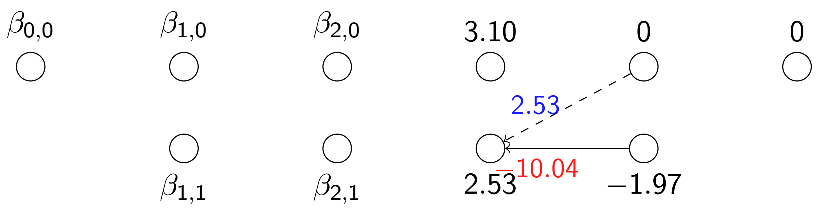
MAP algorithm
Example:Solution
Backward calculation

Example: Backward calculation: Stage 4 to 3



1. States backward probabilities
   $\beta_{3,0} = \log\left(e^{\mu_{3,0,0}+\beta_{4,0}} + e^{\mu_{3,0,1}+\beta_{4,1}}\right) =$
   $\log(e^{3.1+0} + e^{-4.54+(-1.97)}) = 3.10$
   $\beta_{3,1} = \log\left(e^{\mu_{3,1,0}+\beta_{4,0}} + e^{\mu_{3,1,1}+\beta_{4,1}}\right) =$
   $\log(e^{2.53+0} + e^{-10.04+(-1.97)}) = 2.53$
2. $\beta_{4,0} = 3.10 - (3.10) = 0,\ \beta_{4,1} = 2.53 - (3.10) = -0.57$

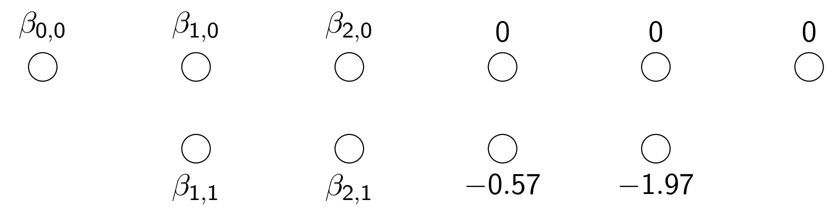
Example: Backward calculation: Stage 4 to 3



1. States backward probabilities
   $\beta_{3,0} = \log\left(e^{\mu_{3,0,0}+\beta_{4,0}} + e^{\mu_{3,0,1}+\beta_{4,1}}\right) = \log(e^{3.1+0} + e^{-4.54+(-1.97)}) = 3.10$
   $\beta_{3,1} = \log\left(e^{\mu_{3,1,0}+\beta_{4,0}} + e^{\mu_{3,1,1}+\beta_{4,1}}\right) = \log(e^{2.53+0} + e^{-10.04+(-1.97)}) = 2.53$
2. $\beta_{4,0} = 3.10 - (3.10) = 0,\ \beta_{4,1} = 2.53 - (3.10) = -0.57$

MAP algorithm
Example:Solution
Backward calculation

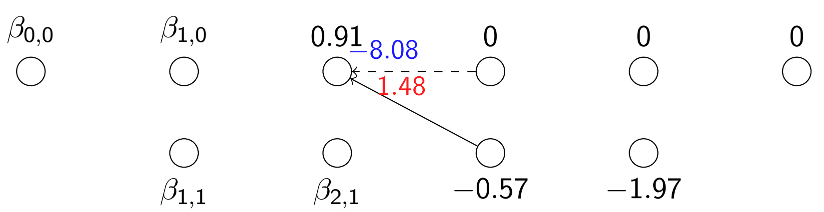
Example: Backward calculation: Stage 4 to 3



1. States backward probabilities
   $\beta_{3,0} = \log\left(e^{\mu_{3,0,0}+\beta_{4,0}} + e^{\mu_{3,0,1}+\beta_{4,1}}\right) = \log(e^{3.1+0} + e^{-4.54+(-1.97)}) = 3.10$
   $\beta_{3,1} = \log\left(e^{\mu_{3,1,0}+\beta_{4,0}} + e^{\mu_{3,1,1}+\beta_{4,1}}\right) = \log(e^{2.53+0} + e^{-10.04+(-1.97)}) = 2.53$
2. $\beta_{4,0} = 3.10 - (3.10) = 0,\ \beta_{4,1} = 2.53 - (3.10) = -0.57$

MAP algorithm
Example:Solution
Backward calculation

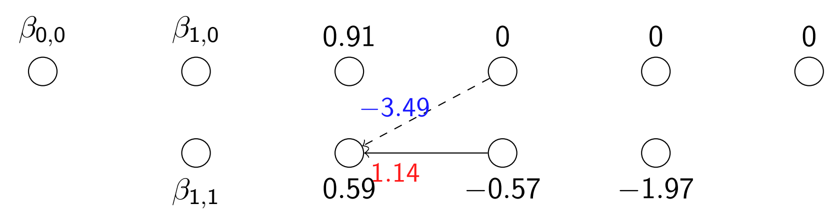
Example: Backward calculation: Stage 3 to 2



1. States backward probabilities
   $\beta_{2,0} = \log\left(e^{\mu_{3,0,0}+\beta_{3,0}} + e^{\mu_{3,0,1}+\beta_{3,1}}\right) = \log(e^{-8.08+0} + e^{1.48+(-0.57)}) = 0.91$
   $\beta_{2,1} = \log\left(e^{\mu_{3,1,0}+\beta_{3,0}} + e^{\mu_{3,1,1}\beta_{3,1}}\right) = \log(e^{-3.49+0} + e^{1.14+(-0.57)}) = 0.59$
2. $\beta_{2,0} = 0.91 - (0.91) = 0,\ \beta_{2,1} = 0.59 - (0.91) = -0.32$

MAP algorithm
Example:Solution
Backward calculation

Example: Backward calculation: Stage 3 to 2

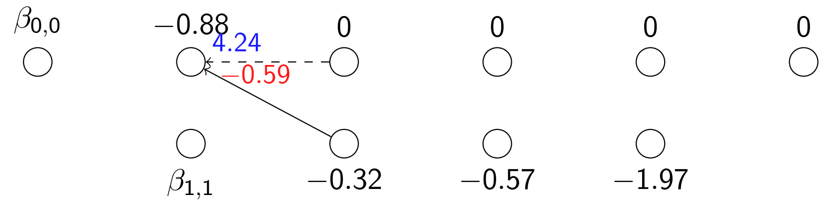


1. States backward probabilities
   $\beta_{2,0} = \log\left(e^{\mu_{3,0,0}+\beta_{3,0}} + e^{\mu_{3,0,1}+\beta_{3,1}}\right) = \log(e^{-8.08+0} + e^{1.48+(-0.57)}) = 0.91$
   $\beta_{2,1} = \log\left(e^{\mu_{3,1,0}+\beta_{3,0}} + e^{\mu_{3,1,1}\beta_{3,1}}\right) = \log(e^{-3.49+0} + e^{1.14+(-0.57)}) = 0.59$
2. $\beta_{2,0} = 0.91 - (0.91) = 0,\ \beta_{2,1} = 0.59 - (0.91) = -0.32$

Example: Backward calculation: Stage 3 to 2

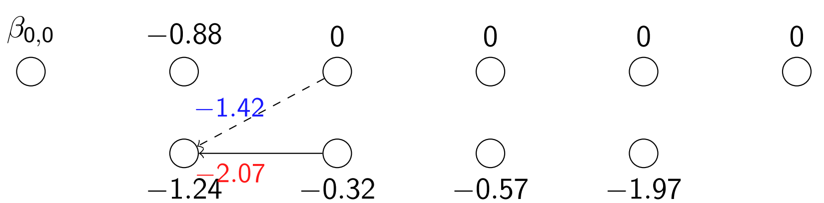
1. States backward probabilities
   $\beta_{2,0} = \log\left(e^{\mu_{3,0,0}+\beta_{3,0}} + e^{\mu_{3,0,1}+\beta_{3,1}}\right) = \log(e^{-8.08+0} + e^{1.48+(-0.57)}) = 0.91$
   $\beta_{2,1} = \log\left(e^{\mu_{3,1,0}+\beta_{3,0}} + e^{\mu_{3,1,1}\beta_{3,1}}\right) = \log(e^{-3.49+0} + e^{1.14+(-0.57)}) = 0.59$
2. $\beta_{2,0} = 0.91 - (0.91) = 0$, $\beta_{2,1} = 0.59 - (0.91) = -0.32$

Example: Backward calculation: Stage 3 to 2



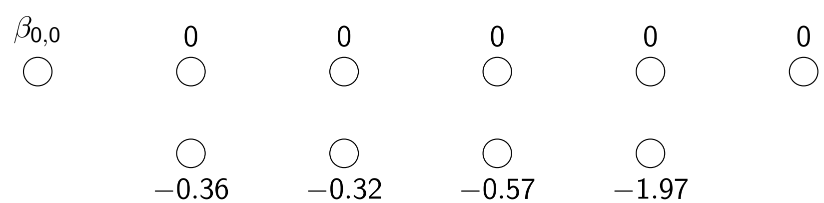
1. States backward probabilities
   $\beta_{1,0} = \log\left(e^{\mu_{2,0,0}+\beta_{2,0}} + e^{\mu_{2,0,1}+\beta_{2,1}}\right) = \log(e^{-4.24+0} + e^{-0.59+(-0.32)}) = -0.88$
   $\beta_{1,1} = \log\left(e^{\mu_{2,1,0}\beta_{2,0}} + e^{\mu_{2,1,1}+\beta_{2,1}}\right) = \log(e^{-1.42+0} + e^{-2.7+(-0.32)}) = -1.24$
2. $\beta_{1,0} = -0.88 - (-0.88) = 0$,
   $\beta_{1,1} = -1.24 - (-0.88) = -0.36$

Example: Backward calculation: Stage 3 to 2



1. States backward probabilities
   $\beta_{1,0} = \log\left(e^{\mu_{2,0,0}+\beta_{2,0}} + e^{\mu_{2,0,1}+\beta_{2,1}}\right) = \log(e^{-4.24+0} + e^{-0.59+(-0.32)}) = -0.88$
   $\beta_{1,1} = \log\left(e^{\mu_{2,1,0}\beta_{2,0}} + e^{\mu_{2,1,1}+\beta_{2,1}}\right) = \log(e^{-1.42+0} + e^{-2.7+(-0.32)}) = -1.24$
2. $\beta_{1,0} = -0.88 - (-0.88) = 0$,
   $\beta_{1,1} = -1.24 - (-0.88) = -0.36$

Example: Backward calculation: Stage 3 to 2

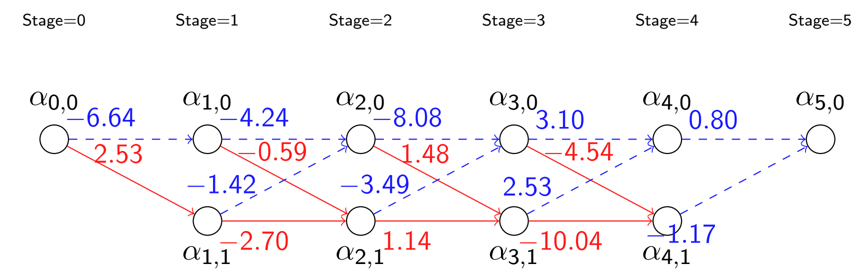


1. States backward probabilities
   $\beta_{1,0} = \log\left(e^{\mu_{2,0,0}+\beta_{2,0}} + e^{\mu_{2,0,1}+\beta_{2,1}}\right) = \log(e^{-4.24+0} + e^{-0.59+(-0.32)}) = -0.88$
   $\beta_{1,1} = \log\left(e^{\mu_{2,1,0}\beta_{2,0}} + e^{\mu_{2,1,1}+\beta_{2,1}}\right) = \log(e^{-1.42+0} + e^{-2.7+(-0.32)}) = -1.24$
2. $\beta_{1,0} = -0.88 - (-0.88) = 0$,
   $\beta_{1,1} = -1.24 - (-0.88) = -0.36$

Example: Forward calculation

- start from the beginning of the trellis and calcualate the sum of the probabilitites
- Init the probability of the first state at the stage to 1 (in log domain to 0): $\alpha_{0,0} = 0$
  1. By using the state probabilities k and the probabilities on the transitions calculate the probabilities for the states $k+1$
     $\alpha_{k,i} = \log\left(\sum_j e^{\mu_{k,i,j}+\alpha_{k-1,j}}\right)$
     (If the values are in the log domain for the sum you have to convert them back to the probability domain.)
  2. Normilize (needed for the numerical stability)
     $\alpha_{k,\#} = \alpha_{k,\#} - \max\left(\alpha_{k,\#}\right)$

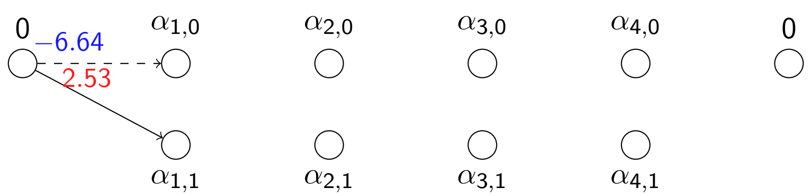
Example: Forward calculation, Forward trellis



$\alpha_{stage,state}$

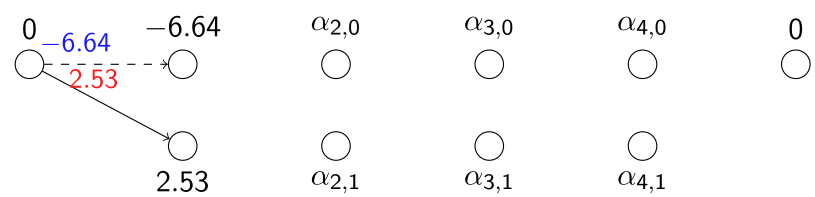
$\alpha_{k,i} = \log\left(\sum_j e^{\mu_{k,i,j}+\alpha_{k-1,j}}\right)$

Example: Forward calculation: Stage 0 to 1



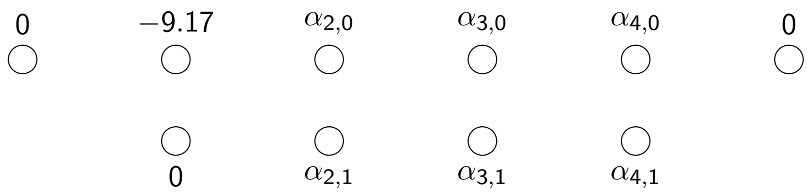
1. The new state probabilities
   $\alpha_{1,0} = \mu_{1,0,0} + \alpha_{0,0} = -6.64$
   $\alpha_{1,1} = \mu_{1,1,0} + \alpha_{0,0} = 2.53$
2. $\alpha_{1,0} = -6.64 - 2.53 = -9.17$, $\alpha_{1,1} = 2.53 - 2.53 = 0$

Example: Forward calculation: Stage 0 to 1



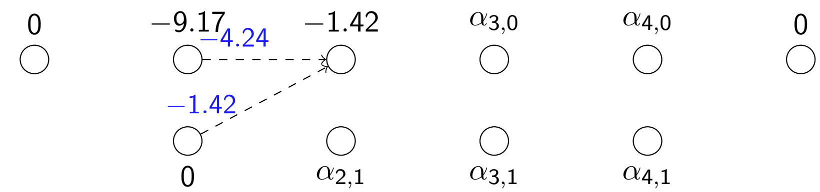
1. The new state probabilities
   $\alpha_{1,0} = \mu_{1,0,0} + \alpha_{0,0} = -6.64$
   $\alpha_{1,1} = \mu_{1,1,0} + \alpha_{0,0} = 2.53$
2. $\alpha_{1,0} = -6.64 - 2.53 = -9.17$, $\alpha_{1,1} = 2.53 - 2.53 = 0$

Example: Forward calculation: Stage 0 to 1



1. The new state probabilities
   $\alpha_{1,0} = \mu_{1,0,0} + \alpha_{0,0} = -6.64$
   $\alpha_{1,1} = \mu_{1,1,0} + \alpha_{0,0} = 2.53$
2. $\alpha_{1,0} = -6.64 - 2.53 = -9.17$, $\alpha_{1,1} = 2.53 - 2.53 = 0$

Example: Forward calculation: Stage 1 to 2

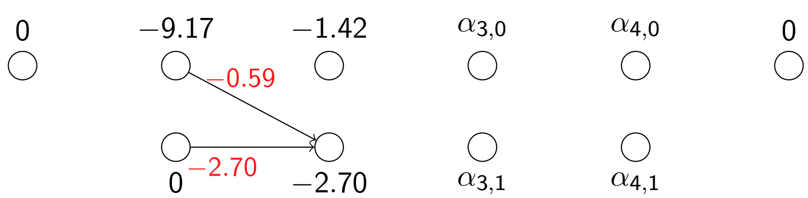


1.

$$\begin{aligned}\alpha_{2,0} &= \log\left(e^{\mu_{2,0,0}+\alpha_{1,0}} + e^{\mu_{2,0,1}+\alpha_{1,1}}\right)\\ &= \log(e^{-4.24+(-9.17)} + e^{-1.42+0}) = -1.42\\ \alpha_{2,1} &= \log\left(e^{\mu_{2,1,0}+\alpha_{1,0}} + e^{\mu_{2,1,1}+\alpha_{1,1}}\right)\\ &= \log(e^{1.48+(-9.17)} + e^{-1.14+0}) = -2.70\end{aligned}$$

2. $\alpha_{2,0} = -1.42 - (-1.42) = 0$, $\alpha_{2,1} = -2.70 - (-1.42) = -1.28$

Example: Forward calculation: Stage 1 to 2

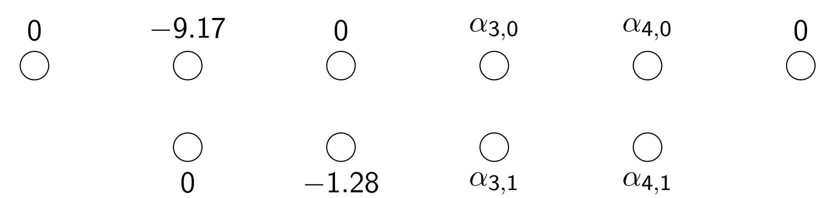


1.

$$\begin{aligned}\alpha_{2,0} &= \log\left(e^{\mu_{2,0,0}+\alpha_{1,0}} + e^{\mu_{2,0,1}+\alpha_{1,1}}\right)\\ &= \log(e^{-4.24+(-9.17)} + e^{-1.42+0}) = -1.42\\ \alpha_{2,1} &= \log\left(e^{\mu_{2,1,0}+\alpha_{1,0}} + e^{\mu_{2,1,1}+\alpha_{1,1}}\right)\\ &= \log(e^{1.48+(-9.17)} + e^{-1.14+0}) = -2.70\end{aligned}$$

2. $\alpha_{2,0} = -1.42 - (-1.42) = 0$, $\alpha_{2,1} = -2.70 - (-1.42) = -1.28$

Example: Forward calculation: Stage 1 to 2

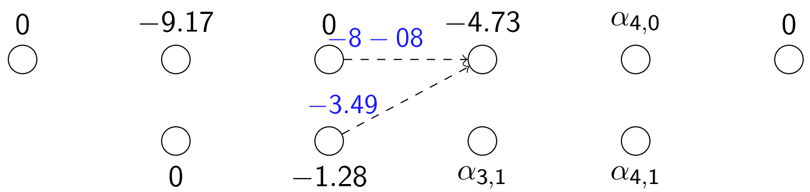


1.

$$\begin{aligned}\alpha_{2,0} &= \log\left(e^{\mu_{2,0,0}+\alpha_{1,0}} + e^{\mu_{2,0,1}+\alpha_{1,1}}\right)\\ &= \log(e^{-4.24+(-9.17)} + e^{-1.42+0}) = -1.42\\ \alpha_{2,1} &= \log\left(e^{\mu_{2,1,0}+\alpha_{1,0}} + e^{\mu_{2,1,1}+\alpha_{1,1}}\right)\\ &= \log(e^{1.48+(-9.17)} + e^{-1.14+0}) = -2.70\end{aligned}$$

2. $\alpha_{2,0} = -1.42 - (-1.42) = 0$, $\alpha_{2,1} = -2.70 - (-1.42) = -1.28$

Example: Forward calculation: Stage 2 to 3

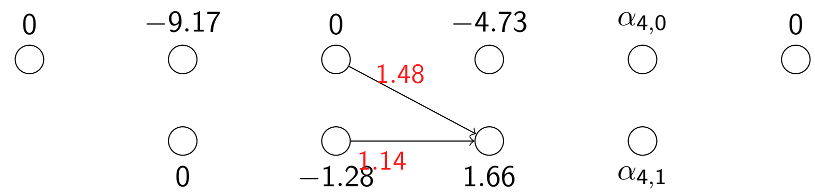


1. $\alpha_{3,0} = \log\left(e^{\mu_{3,0,0}+\alpha_{2,0}} + e^{\mu_{3,0,1}+\alpha_{2,1}}\right) = \log(e^{-8.08+0} + e^{-3.49+(-1.28)}) = -4.73$

   $\alpha_{3,1} = \log\left(e^{\mu_{3,1,0}+\alpha_{2,0}} + e^{\mu_{3,1,1}+\alpha_{2,1}}\right) = \log(e^{-4.54+0} + e^{-10.04+(-1.28)}) = 1.66$

2. $\alpha_{3,0} = -4.73 - 1.66 = -6.40$, $\alpha_{3,1} = 1.66 - 1.66 = 0$

Example: Forward calculation: Stage 2 to 3

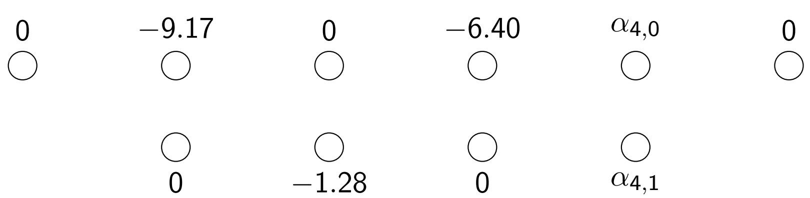


1. $\alpha_{3,0} = \log\left(e^{\mu_{3,0,0}+\alpha_{2,0}} + e^{\mu_{3,0,1}+\alpha_{2,1}}\right) = \log(e^{-8.08+0} + e^{-3.49+(-1.28)}) = -4.73$

   $\alpha_{3,1} = \log\left(e^{\mu_{3,1,0}+\alpha_{2,0}} + e^{\mu_{3,1,1}+\alpha_{2,1}}\right) = \log(e^{-4.54+0} + e^{-10.04+(-1.28)}) = 1.66$

2. $\alpha_{3,0} = -4.73 - 1.66 = -6.40$, $\alpha_{3,1} = 1.66 - 1.66 = 0$

Example: Forward calculation: Stage 2 to 3

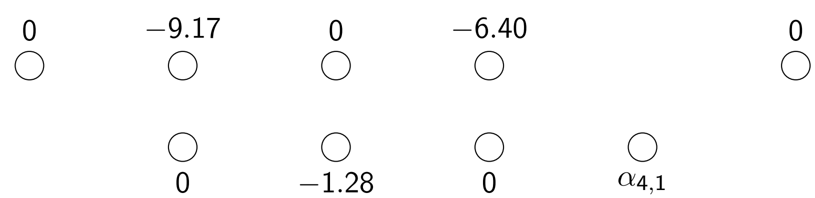


1. $\alpha_{3,0} = \log\left(e^{\mu_{3,0,0}+\alpha_{2,0}} + e^{\mu_{3,0,1}+\alpha_{2,1}}\right) = \log(e^{-8.08+0} + e^{-3.49+(-1.28)}) = -4.73$

   $\alpha_{3,1} = \log\left(e^{\mu_{3,1,0}+\alpha_{2,0}} + e^{\mu_{3,1,1}+\alpha_{2,1}}\right) = \log(e^{-4.54+0} + e^{-10.04+(-1.28)}) = 1.66$

2. $\alpha_{3,0} = -4.73 - 1.66 = -6.40$, $\alpha_{3,1} = 1.66 - 1.66 = 0$

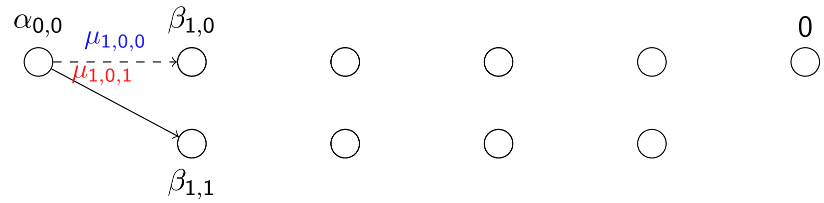
Example: Forward calculation:



Bite aposteriori Pb

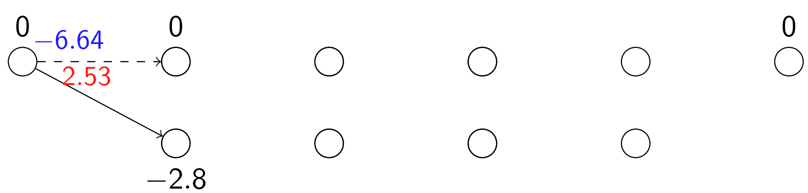
- The bit aposteriori probability is calculated by combing forward probability is states k backward probabilty in states $k+1$ and transitions probabilities.
- $\log \frac{p(y_k|x_k=1,x)}{p(y_k|x_k=0,x)} \frac{\sum\limits_{x_k=1} e^{\alpha_{k,i}+\mu_{k,i,j}+\beta_{k+1,j}}}{\sum\limits_{x_k=0} e^{\alpha_{k,i}+\mu_{k,i,j}+\beta_{k+1,j}}}$

Example: a-posteriori for the bits: bit 1



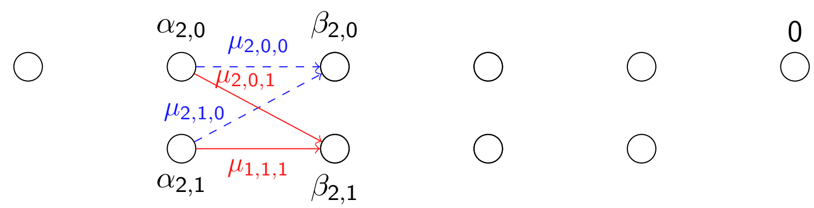
$$\log\left(\frac{e^{\alpha_{0,0}+\mu_{1,0,1}+\beta_{1,1}}}{e^{\alpha_{0,0}+\mu_{1,0,0}+\beta_{1,0}}}\right) = \log\left(\frac{e^{0-2.53+(-0.36)}}{e^{0-6.63+0}}\right)$$
$$= 2.17-(-6.64) = 8.81$$

Example: a-posteriori for the bits: bit 1



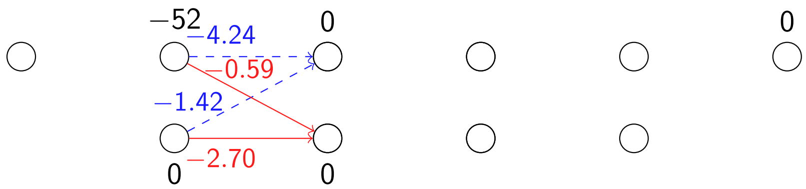
$$\log\left(\frac{e^{\alpha_{0,0}+\mu_{1,0,1}+\beta_{1,1}}}{e^{\alpha_{0,0}+\mu_{1,0,0}+\beta_{1,0}}}\right) = \log\left(\frac{e^{0-2.53+(-0.36)}}{e^{0-6.63+0}}\right)$$
$$= 2.17-(-6.64) = 8.81$$

Example: a-posteriori for the bits: bit 2



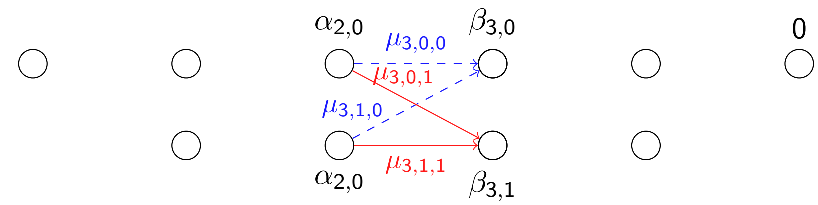
$$\log\left(\frac{e^{\alpha_{1,0}+\mu_{2,0,1}+\beta_{2,1}}+e^{\alpha_{1,1}+\mu_{1,1,1}+\beta_{2,1}}}{e^{\alpha_{1,0}+\mu_{2,0,0}+\beta_{2,0}}+e^{\alpha_{1,1}+\mu_{2,1,0}+\beta_{2,0}}}\right)$$
$$= \log\left(e^{-9.17-4.24+0}+e^{0-1.42+0}\right) - \log\left(e^{-9.17-0.59-0.32}+e^{0-2.70-0.32}\right)$$
$$= -3.14$$

Example: a-posteriori for the bits: bit 2



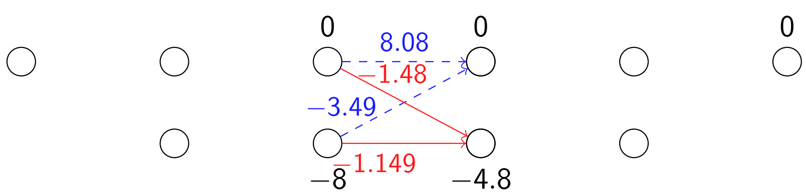
$$\log\left(\frac{e^{\alpha_{1,0}+\mu_{2,0,1}+\beta_{2,1}}+e^{\alpha_{1,1}+\mu_{1,1,1}+\beta_{2,1}}}{e^{\alpha_{1,0}+\mu_{2,0,0}+\beta_{2,0}}+e^{\alpha_{1,1}+\mu_{2,1,0}+\beta_{2,0}}}\right)$$
$$=\log\left(e^{-9.17-4.24+0}+e^{0-1.42+0}\right)-\log\left(e^{-9.17-0.59-0.32}+e^{0-2.70-0.32}\right)$$
$$=-3.14$$

Example: a-posteriori for the bits: bit 3



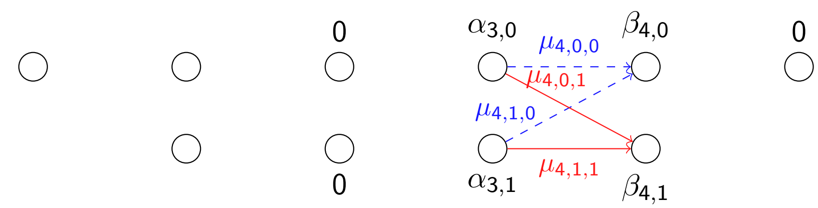
$$\log\left(\frac{e^{\alpha_{2,0}+\mu_{3,0,1}+\beta_{3,1}}+e^{\alpha_{2,1}+\mu_{3,1,1}+\beta_{3,1}}}{e^{\alpha_{2,0}+\mu_{3,0,0}+\beta_{3,0}}+e^{\alpha_{2,1}+\mu_{3,1,0}+\beta_{3,0}}}\right)$$
$$=\log\left(e^{0+8.08+0}+e^{-1.28-3.49-0}\right)-\log\left(e^{0+1.47-0.57}+e^{-1.28-1.14-0.57}\right)$$
$$=5.64$$

Example: a-posteriori for the bits: bit 3



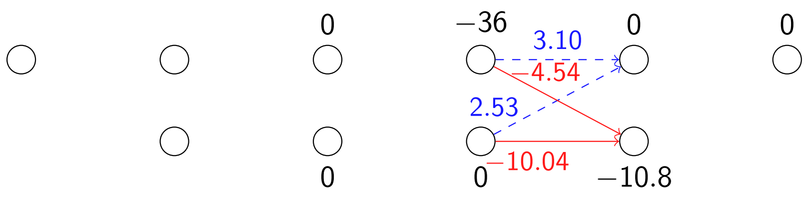
$$\log\left(\frac{e^{\alpha_{2,0}+\mu_{3,0,1}+\beta_{3,1}}+e^{\alpha_{2,1}+\mu_{3,1,1}+\beta_{3,1}}}{e^{\alpha_{2,0}+\mu_{3,0,0}+\beta_{3,0}}+e^{\alpha_{2,1}+\mu_{3,1,0}+\beta_{3,0}}}\right)$$
$$=\log\left(e^{0+8.08+0}+e^{-1.28-3.49-0}\right)-\log\left(e^{0+1.47-0.57}+e^{-1.28-1.14-0.57}\right)$$
$$=5.64$$

Example: a-posteriori for the bits: bit 4



$$\log\left(\frac{e^{\alpha_{3,0}+\mu_{4,0,1}+\beta_{4,1}}+e^{\alpha_{3,1}+\mu_{4,1,1}+\beta_{4,1}}}{e^{\alpha_{3,0}+\mu_{4,0,0}+\beta_{4,0}}+e^{\alpha_{3,1}+\mu_{4,1,0}+\beta_{4,0}}}\right)$$
$$=\log\left(e^{-6.40+3.10+0}+e^{0+2.53+-1.97}\right)-\log\left(e^{-6.40-4.54+0}+e^{0-10.04-1.97}\right)$$
$$=-1.41$$

Example: a-posteriori for the bits: bit 4



$$\log\left(\frac{e^{\alpha_{3,0}+\mu_{4,0,1}+\beta_{4,1}}+e^{\alpha_{3,1}+\mu_{4,1,1}+\beta_{4,1}}}{e^{\alpha_{3,0}+\mu_{4,0,0}+\beta_{4,0}}+e^{\alpha_{3,1}+\mu_{4,1,0}+\beta_{4,0}}}\right)$$

$$= \log\left(e^{-6.40+3.10+0}+e^{0+2.53+-1.97}\right)-\log\left(e^{-6.40-4.54+0}+e^{0-10.04-1.97}\right)$$

$$= -1.41$$