S72.3280

Tutorial 4

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Exercise 2

We have two variables for one of them the loglikelihood ratio is given to be

$$L_{c_1} = -2$$

Other variable has a loglikelihood values

$$L_{c_2} = \{-5 \quad -3 \quad -2.5 \quad -2 \quad -1.5 \quad -1 \quad 0\}$$

Calculate how well the max operation approximates the exact sum used in logMAP operation.

Solution 2

LogMAP calculates

$$\ln\left(e^{L_1} + e^{L_2}\right) =$$

$$|L_1| > |L_2| \Rightarrow \ln\left(e^{L_1}\left(1 + e^{-|L_2 - L_1|}\right)\right) = L_1 + \ln\left(1 + e^{-|L_2 - L_1|}\right)$$

$$|L_2| > |L_1| \Rightarrow \ln\left(e^{L_2}\left(1 + e^{-|L_1 - L_2|}\right)\right) = L_2 + \ln\left(1 + e^{-|L_2 - L_1|}\right)$$

$$\max{}^*(L_1, L_2) = \max(L_1, L_2) + \ln\left(1 + e^{-|L_2 - L_1|}\right)$$

The term $\ln\left(1 + e^{-|L_2 - L_1|}\right)$ is called correction term and dropped in maxlogMAP alogorithm. MaxlogMAP uses approximation

$$\max{}^*(L_1, L_2) \approx \max(L_1, L_2)$$

L2	LogMAP	Correction $\ln\left(1 + e^{-\lvert L_2 - L_1\rvert}\right)$	maxLogMAP
-10	-1.9997	0.0003	-2
-5	-1.95	0.049	-2
-3	-1.69	0.31	-2
-2.5	-1.53	0.47	-2
-2	-1.31	0.69	-2
-1.5	-1.02	0.47	-1.5
-1	-0.69	0.31	-1
0	0.13	0.12	0
1	1.04	0.049	1

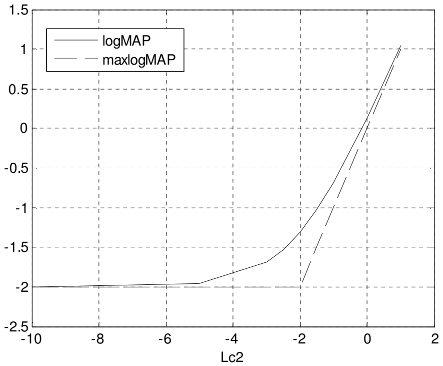

Figure. Numerical results to the logMAP and maxLogMAP operations

Exercise 3

The code is generated accordingly to the trellis in the figure below. We want to calculate

1. loglikelihood for the input bit marginal probability.
2. loglikelihood probabilities for the symbol *c1*.
3. loglikelihood probabilities for the symbol *c2*.

Use for the calculations maxlogMAP and logMAP alghorithm.
For a trellis section in figure we know the initial state probabilities
A the final state probabilities B, the extrinsic information for the bit Le and probabilities for the received symbols *c1* and *c2*.

$$A = [-0.7\ \ 0\ \ -0.8\ \ -5.1]$$

$$B = [0\ \ -3.2\ \ -0.9\ \ -5.1]$$

$$L_e = 5$$

$$L_{c1} = -1.1$$

$$L_{c1} = 2.8$$

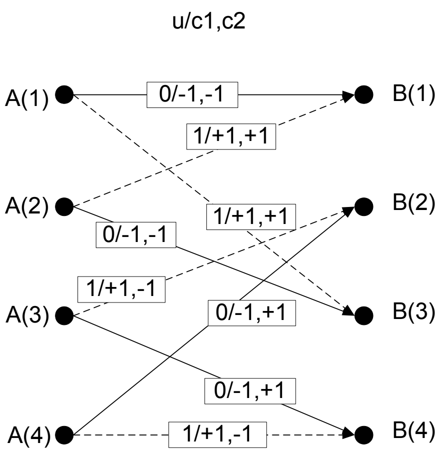

Figure. The trellis section

Solution 3

For calculating a marginal probability we have to calculate each path probability for each transition in the trellis. The marginal probabilities are calculated by summing over all the paths in the trellis where the particular variable is 1 and over all the transitions in the trellis where the variable is 0.
The loglikelihood ratio of the marginal probability is the ratio of these two marginal probilities.
For example for the input bit u the marginal probability is calculated as

$$L(u) = \ln \frac{\sum_{s:u=1} A(s)M(s)B(s)}{\sum_{s:u=0} A(s)M(s)B(s)}$$

$$= \ln \frac{\left\{ \begin{array}{l} A(1)M\left(1/1,1\right)B(3) + A(2)M\left(1/1,1\right)B(1) + \\ A(3)M\left(1/1,-1\right)B(2) + A(4)M\left(1/1,-1\right)B\left(4\right) \end{array} \right\}}{\left\{ \begin{array}{l} A(1)M\left(0/-1,-1\right)B(1) + A(2)M\left(0/-1,-1\right)B(3) + \\ A(3)M\left(0/-1,1\right)B(4) + A(4)M\left(0/-1,1\right)B\left(2\right) \end{array} \right\}}$$

Each path in the trellis has weight accordingly to the probabilities of the symbols defined on this particular path.

$$M\left(u/c_1,c_2\right) = P\left(c_1 \mid u_i\right)P\left(c_2 \mid u_i\right)P\left(u_e \mid u_i\right)$$

where

$P\left(c_1 \mid u_i\right)$ probability of occurrence of the first symbol *c1*

$P\left(c_2 \mid u_i\right)$ probability of occurrence of the second symbol *c2*

$P\left(u_e \mid u_i\right)$ extrinsic probability for the symbol in the particular path.

The probabilities can be calculated from the loglikelihood ratios of the particular variables. Alternatively we can operate in the logarithmic domain and replace the probabilities by their logarithms.
In logarithmic domain the probability that a variable v is one is expressed as

$$L\left(v=1\right) = L_v - \ln\left(1+e^{L_v}\right)$$

and that it is zeros is

$$L\left(v=0\right) = 0 - \ln\left(1+e^{L_v}\right)$$

By using this properties, for the first transition in the trellis

$$\ln\left(M\left(0/-1,-1\right)\right) = 0 - \ln\left(1+e^{L_e}\right) + 0 - \ln\left(1+e^{L_{c1}}\right) + 0 - \ln\left(1+e^{L_{c2}}\right)$$

We have in total four types of different transitions

$$\ln\left(M\left(0/-1,1\right)\right) = 0 - \ln\left(1+e^{L_e}\right) + 0 - \ln\left(1+e^{L_{c1}}\right) + L_{c2} - \ln\left(1+e^{L_{c2}}\right)$$

$$\ln\left(M\left(1/1,1\right)\right) = L_e - \ln\left(1+e^{L_e}\right) + L_{c1} - \ln\left(1+e^{L_{c1}}\right) + L_{c2} - \ln\left(1+e^{L_{c2}}\right)$$

$$\ln\left(M\left(1/1,-1\right)\right) = L_e - \ln\left(1+e^{L_e}\right) + L_{c1} - \ln\left(1+e^{L_{c1}}\right) + 0 - \ln\left(1+e^{L_{c2}}\right)$$

We see that all of the transitions have added a scaling term

$$-\ln\left(1+e^{L_e}\right) - \ln\left(1+e^{L_{c1}}\right) - \ln\left(1+e^{L_{c2}}\right)$$

Since this term is common for all the code words (it is in all the transitions) it does not impact our decision that operates on the differences between the codeword probabilities (in log domain). In result of that we can drop it from our further calculations.

By combining the transitions and state probabilities we get for the first transition in the trellis

$$\ln\left(A(1)M\left(0/-1,-1\right)B(1)\right) = L_{A(1)} + \ln M\left(0/-1,-1\right) + L_{B(1)} = L_{A(1)} + L_{B(1)}$$

When we sum together the probabilities in different paths through the trellis we have to operate in the probability domain. For this particular transition we get

$$e^{\ln\left(A(1)M\left(0/-1,-1\right)B(1)\right)} = e^{L_{A(1)}+L_{B(1)}}$$

The marginal probability for the symbol u becomes

$$L(u) = \ln \frac{\sum_{s:u=1} A(s)M(s)B(s)}{\sum_{s:u=0} A(s)M(s)B(s)}$$

$$= \ln \frac{\left\{ \begin{array}{l} \exp\left(A(1)M\left(1/1,1\right)B(3)\right) + \exp\left(A(2)M\left(1/1,1\right)B(1)\right) + \\ \exp\left(A(3)M\left(1/1,-1\right)B(2)\right) + \exp\left(A(4)M\left(1/1,-1\right)B\left(4\right)\right) \end{array} \right\}}{\left\{ \begin{array}{l} \exp\left(A(1)M\left(0/-1,-1\right)B(1)\right) + \exp\left(A(2)M\left(0/-1,-1\right)B(3)\right) + \\ \exp\left(A(3)M\left(0/-1,1\right)B(4)\right) + \exp\left(A(4)M\left(0/-1,1\right)B\left(2\right)\right) \end{array} \right\}}$$

$$\begin{aligned} L(u) &= \ln\left(e^{L_{A(1)}+L_e+L_{c1}+L_{c2}+L_{B(3)}} + e^{L_{A(2)}+L_e+L_{c1}+L_{c2}+L_{B(1)}} + e^{L_{A(3)}+L_e+L_{c1}+L_{B(2)}} + e^{L_{A(4)}+L_e+L_{c1}+L_{B(4)}}\right) \\ &\quad -\ln\left(e^{L_{A(1)}+L_{B(1)}} + e^{L_{A(2)}+L_{B(3)}} + e^{L_{A(3)}+L_{c2}+L_{B(4)}} + e^{L_{A(1)}+L_{c2}+L_{B(2)}}\right) \\ &= L_e + L_{c1} + \ln\left(e^{L_{A(1)}+L_{c2}+L_{B(3)}} + e^{L_{A(2)}+L_{c2}+L_{B(1)}} + e^{L_{A(3)}+L_{B(2)}} + e^{L_{A(4)}+L_{B(4)}}\right) \\ &\quad -\ln\left(e^{L_{A(1)}+L_{B(1)}} + e^{L_{A(2)}+L_{B(3)}} + e^{L_{A(3)}+L_{c2}+L_{B(4)}} + e^{L_{A(1)}+L_{c2}+L_{B(2)}}\right) \end{aligned}$$

Where we see that we could separate the extrinsic information Le and the systematic bit information $Lc1$.

Similarly as for the bit u also the marginal probabilities of the coded bits c1 and c2 are calculated by summing over the all the transitions where they are 1 and where they are 0.
For c1 it is similar sum as for bit u

$$L(c_1) = \ln \frac{\sum_{s:c1=1} A(s)M(s)B(s)}{\sum_{s:c1=0} A(s)M(s)B(s)}$$

$$= \ln \frac{\left\{ \begin{array}{l} \exp\left(A(1)M\left(1/1,1\right)B(3)\right) + \exp\left(A(2)M\left(1/1,1\right)B(1)\right) + \\ \exp\left(A(3)M\left(1/1,-1\right)B(2)\right) + \exp\left(A(4)M\left(1/1,-1\right)B\left(4\right)\right) \end{array} \right\}}{\left\{ \begin{array}{l} \exp\left(A(1)M\left(0/-1,-1\right)B(1)\right) + \exp\left(A(2)M\left(0/-1,-1\right)B(3)\right) + \\ \exp\left(A(3)M\left(0/-1,1\right)B(4)\right) + \exp\left(A(4)M\left(0/-1,1\right)B\left(2\right)\right) \end{array} \right\}}$$

and the calculation is

$$\begin{aligned} L(c_1) &= \ln\left(e^{L_{A(1)}+L_e+L_{c1}+L_{c2}+L_{B(3)}} + e^{L_{A(2)}+L_e+L_{c1}+L_{c2}+L_{B(1)}} + e^{L_{A(3)}+L_e+L_{c1}+L_{B(2)}} + e^{L_{A(4)}+L_e+L_{c1}+L_{B(4)}}\right) \\ &\quad -\ln\left(e^{L_{A(1)}+L_{B(1)}} + e^{L_{A(2)}+L_{B(3)}} + e^{L_{A(3)}+L_{c2}+L_{B(4)}} + e^{L_{A(1)}+L_{c2}+L_{B(2)}}\right) \\ &= L_e + L_{c1} + \ln\left(e^{L_{A(1)}+L_{c2}+L_{B(3)}} + e^{L_{A(2)}+L_{c2}+L_{B(1)}} + e^{L_{A(3)}+L_{B(2)}} + e^{L_{A(4)}+L_{B(4)}}\right) \\ &\quad -\ln\left(e^{L_{A(1)}+L_{B(1)}} + e^{L_{A(2)}+L_{B(3)}} + e^{L_{A(3)}+L_{c2}+L_{B(4)}} + e^{L_{A(1)}+L_{c2}+L_{B(2)}}\right) \end{aligned}$$

For the coded bit c2 we end up with the equations

$$L(c_2) = \ln \frac{\sum_{s:c2=1} A(s)M(s)B(s)}{\sum_{s:c2=0} A(s)M(s)B(s)}$$

$$= \ln \frac{\left\{ \begin{array}{l} \exp\left(A(1)M\left(1/1,1\right)B(3)\right) + \exp\left(A(2)M\left(1/1,1\right)B(1)\right) + \\ \exp\left(A(3)M\left(0/-1,1\right)B(4)\right) + \exp\left(A(4)M\left(0/-1,1\right)B\left(2\right)\right) \end{array} \right\}}{\left\{ \begin{array}{l} \exp\left(A(1)M\left(0/-1,-1\right)B(1)\right) + \exp\left(A(2)M\left(0/-1,-1\right)B(3)\right) + \\ \exp\left(A(3)M\left(1/1,-1\right)B(2)\right) + \exp\left(A(4)M\left(1/1,-1\right)B\left(4\right)\right) \end{array} \right\}}$$

Notice that compared to the bit c1 marginal probability the paths from the states A(3) and A(4) are allocated differently. This is because compared to bit c1 the bit c2 takes alternative values.

$$\begin{aligned} L(c_1) &= \ln\left(\mathrm{e}^{L_{A(1)}+L_e+L_{c1}+L_{c2}+L_{B(3)}} + \mathrm{e}^{L_{A(2)}+L_e+L_{c1}+L_{c2}+L_{B(1)}} + \mathrm{e}^{L_{A(3)}+L_{c2}+L_{B(4)}} + \mathrm{e}^{L_{A(4)}+L_{c2}+L_{B(2)}}\right) \\ &\quad - \ln\left(\mathrm{e}^{L_{A(1)}+L_{B(1)}} + \mathrm{e}^{L_{A(2)}+L_{B(3)}} + \mathrm{e}^{L_{A(3)}+L_e+L_{c1}+L_{B(2)}} + \mathrm{e}^{L_{A(1)}+L_e+L_{c1}+L_{B(4)}}\right) \\ &= L_{c2} + \ln\left(\mathrm{e}^{L_{A(1)}+L_e+L_{c1}+L_{B(3)}} + \mathrm{e}^{L_{A(2)}+L_e+L_{c1}+L_{B(1)}} + \mathrm{e}^{L_{A(3)}+L_{B(4)}} + \mathrm{e}^{L_{A(4)}+L_{B(2)}}\right) \\ &\quad - \ln\left(\mathrm{e}^{L_{A(1)}+L_{B(1)}} + \mathrm{e}^{L_{A(2)}+L_{B(3)}} + \mathrm{e}^{L_{A(3)}+L_e+L_{c1}+L_{B(2)}} + \mathrm{e}^{L_{A(1)}+L_e+L_{c1}+L_{B(4)}}\right) \end{aligned}$$

The calculations above are using exact evaluation based on the logMAP approach. The calculations can be simplified by using maxlogMAP. maxlogMAP operation approximates the exact logMAP operation

$$\max{}^*\left(L_1, L_2\right) = \ln\left(e^{L_1} + e^{L_2}\right) = \max\left(L_1, L_2\right) + \ln\left(1 + e^{-|L_2 - L_1|}\right)$$

$$\approx \max\left(L_1, L_2\right)$$

This approximation can be used iteratively on the calculations of the marginals above.

$\ln\left(M\left(0/-1,-1\right)\right)$	0
$\ln\left(M\left(1/1,1\right)\right)$	6.7
$\ln\left(M\left(0/-1,1\right)\right)$	2.8
$\ln\left(M\left(1/1,-1\right)\right)$	3.9

logMAP decoding

bit	$\ln\left(p\left(u=1\right)\right)$	$\ln\left(p\left(u=0\right)\right)$	*logMAP*
u	6.88	-0.049	6.93
s1	6.88	-0.049	6.93
s2	6.88	-0.59	6.29

maxlogMAP decoding

bit	$\ln\left(p\left(u=1\right)\right)$	$\ln\left(p\left(u=0\right)\right)$	*maxlogMAP*
u	6.7	-0.7	7.4
s1	6.7	-0.7	7.4
s2	6.7	-0.1	6.8