Solution 1

The MAP estimation is calculated as:

$$\log \frac{\sum_{x_k=1} p\left(y_1, y_2, \dots, y_k, \dots, y_N \middle| x_1, x_2, \dots, x_k = 1, \dots, x_N\right)}{\sum_{x_k=0} p\left(y_1, y_2, \dots, y_k, \dots, y_N \middle| x_1, x_2, \dots, x_k = 0, \dots, x_N\right)}$$

In our code we have $N_b = 3$ information bits $N_x = 6$ code symbols.
In total we have $2^{N_b} = 8$ codewords.

	b_1		b_2		b_3	
	x_1	x_2	x_3	x_4	x_5	x_6
1	0	0	0	0	0	0
2	0	0	0	0	1	1
3	0	0	1	1	0	1
4	0	0	1	1	1	0
5	1	1	0	1	1	1
6	1	1	0	1	0	0
7	1	1	1	0	1	0
8	1	1	1	0	0	1

Symbol Probabilities

For calculating the marginal proability we have to calculate codeword proabilities.
The codeword proability is multipliclation of individual symbol probabilities.

$$p(y_2|x_1 = X)p(y_2|x_2 = X)p(y_3|x_3 = X)p(y_k|x_4 = X)p(y_5|x_5 = X)p(y_6|x_6 = X)$$

Here we assume that all the prior proabilities for all the symbols are the same. Since we are interested in ratio of the proabilities therefore we can omit them.
For example for the first codeword:

$$p(y_2|x_1 = 0)p(y_2|x_2 = 0)p(y_3|x_3 = 0)p(y_k|x_4 = 0)p(y_5|x_5 = 0)p(y_6|x_6 = 0)$$

Symbol Probability

In a Gaussian noise the scaled received bit corresponds to the loglikelihood ratio of the bit probalities.

$$L_k = \log\left(\frac{p(y_k|x_k = 0)}{p(y_k|x_k = 1)}\right) = rate \cdot \frac{E_b}{N_0} y_k$$

By considerig the probability sums to one:

$$p(y_k|x_k = 1) = 1 - p(y_k|x_k = 0)$$

$$\log\left(\frac{p(y_k|x_k = 0)}{1 - p(y_k|x_k = 0)}\right) = rate \cdot \frac{E_b}{N_0} y_k$$

$$p(y_k|x_k = 0) = \frac{e^{L_k}}{1 + e^{L_k}}$$

$$p(y_k|x_k = 1) = \frac{1}{1 + e^{L_k}}$$

Symbol Probability ...

Because $1 + e^L$ term is same for all the symbols (it is scaling) we can drop also that term.
For convenience it is better to remember and operate with $\log(p(y_k|x_k = 0)) = L_k$

Notce that in this exercise $SNR \cdot rate = 1$.

k	1	2	3	4	5	6
y_k	-1.84	0.12	0.36	1.42	1.17	-1.73
$L_k = SNR \cdot rate \cdot y_k$	-1.84	0.12	0.36	1.42	1.17	-1.73

Codeword probability

Codewords probabilities are multiplied by e^{L_k} if the coded bit is 0 and done nothing if bit is 1.
If operating in log domain the multiplication is replaced by the sum.

codeword	1	2	3	4	5	6	7	8
$\log(probability)$	-0.5	0.06	-0.55	-3.45	0.36	-0.2	-0.31	2.59

Marginal probability

Marginal probability is calculated as the sum over the probabilities of the codewords where this particular bit is one $x_3 = 1$. These are the codewords $2, 4, 5, 7$.
Because we are operating in log domain before we sum over them to be power of e.

$$\begin{aligned} p(b_3 = 1|\mathbf{y}) &\sim \log\left(e^{\log(0.06)} + e^{\log(-3.45)} + e^{\log(0.36)} + e^{\log(-0.31)}\right) \\ &= -3.34 \end{aligned}$$

This result is not exactly probability since we have left out all the scaling component.

We can use it for calculation of loglikelihood ratio.

$$\begin{aligned} p(b_3 = 0|\mathbf{y}) &\sim \log\left(e^{\log(-0.5)} + e^{\log(-0.55)} + e^{\log(-0.2)} + e^{\log(2.59)}\right) \\ &= 1.34 \end{aligned}$$

$$\begin{aligned} \log\left(\frac{p(b_3 = 0|\mathbf{y})}{p(b_3 = 1|\mathbf{y})}\right) &= \log\left(p(b_3 = 0|\mathbf{y})\right) - \log\left(p(b_3 = 1|\mathbf{y})\right) \\ &= 1.34 - (-3.34) = 4.68 \end{aligned}$$

We were following a brute force approach for computing the codeword probabilities.
For computing each codeword proabilitity we had to sum together 6 component symbol probabilities. This is total 5 operation and over all the codewords we have $2^{N_b} \cdot 5$ operations.
For calculating marginal proability we have to calculate sum of the exponentials. Sum of four symbols for $p(b_3 = 0|\mathbf{y})$ and similar sum for $p(b_3 = 1|\mathbf{y})$ total 8 exponents and 6 sums.
Finally we have to substract the logarithms: 2 operations for taking logarithms and 1 for substractions.
In total we have $8 \cdot 5 + 8 + 6 + 2 + 1 = 57$ operations.